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Implications for new physics from fine-tuning arguments
2. Little Higgs models

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Abstract: We examine the fine-tuning associated to electroweak breaking in Little Higgs scenarios and find it to be always substantial and, generically, much higher than suggested by the rough estimates usually made. This is due to implicit tunings between parameters that can be overlooked at first glance but show up in a more systematic analysis. Focusing on four popular and representative Little Higgs scenarios, we find that the fine-tuning is essentially comparable to that of the Little Hierarchy problem of the Standard Model (that these scenarios attempt to solve) and higher than in supersymmetric models. This does not demonstrate that all Little Higgs models are fine-tuned, but stresses the need of a careful analysis of this issue in model-building before claiming that a particular model is not fine-tuned. In this respect we identify the main sources of potential fine-tuning that should be watched out for, in order to construct a successful Little Higgs model, which seems to be a non-trivial goal.

Keywords: Spontaneous Symmetry Breaking, Beyond Standard Model, Higgs Physics.

Contents

1. Introduction

In this paper we continue the exam of the implications for new physics from fine-tuning arguments. In a previous paper [1] we revisited the use of the Big Hierarchy problem of the Standard Model (SM) to estimate the scale of new physics, $\Lambda_{\rm SM}$, illustrating our results with two physically relevant examples: right handed (see-saw) neutrinos and supersymmetry (SUSY). Here we study Little Higgs (LH) scenarios as the new physics beyond the SM.

LH models were introduced as an alternative to SUSY in order to solve the Little Hierarchy problem. Very briefly, the latter consists in the following: in the SM (treated as an effective theory valid below $\Lambda_{\rm SM}$) the mass parameter m^2 in the Higgs potential

$$V = \frac{1}{2}m^2h^2 + \frac{1}{4}\lambda h^4\,, \tag{1.1}$$

receives important quadratically-divergent contributions [2]. At one-loop,

$$\delta_q m^2 = \frac{3}{64\pi^2}(3g^2 + g'^2 + 8\lambda - 8\lambda_t^2)\Lambda_{\rm SM}^2\,, \tag{1.2}$$

where g, g', λ and λ_t are the $SU(2) \times U(1)_Y$ gauge couplings, the quartic Higgs coupling and the top Yukawa coupling, respectively, and Λ_{SM} is a common UV cut-off. The requirement of no fine-tuning between the above contribution and the tree-level value of m^2 sets an upper bound on Λ_{SM}. e.g. for a Higgs mass $m_h = 115 - 200\,\mathrm{GeV}$,

$$\left|\frac{\delta_q m^2}{m^2}\right| \leq 10 \;\Rightarrow\; \Lambda_{SM} \lesssim 2 - 3\,\mathrm{TeV}\,. \tag{1.3}$$

This upper bound on Λ_{SM} is in a certain tension with the experimental lower bounds on the suppression scale Λ of higher order operators, derived from fits to precision electroweak data [3], which typically require $\Lambda \gtrsim 10\,\mathrm{TeV}$; and this is known as the Little Hierarchy problem.

Let us briefly outline the general structure of LH models. Their two basic ingredients are, first, that the SM Higgs is a Goldstone boson of a spontaneously broken global symmetry; and second, the explicit breaking of this symmetry by gauge and Yukawa couplings in a collective way (a coupling alone is not able to produce enough breaking to give a mass to the SM Higgs). In consequence, the SM Higgs is a pseudo-Goldstone boson, with mass protected at 1-loop from quadratically divergent contributions. In principle, this is enough to avoid the Little Hierarchy problem: if the quadratic corrections to m^2 appear at the 2-loop level, the extra $(4\pi)^{-2}$ suppression factor in $\delta_q m^2$ allows for a 10 TeV cut-off with no fine-tuning price.

It should be noticed that the above argument does not imply that in LH models there are no extra states at all below 10 TeV. As discussed in the previous paper [1], even if the quadratic divergences cancel exactly, the new physics states do contribute with logarithmic and finite corrections to m^2, so their masses should still not be larger than 2–3 TeV (as happens in the supersymmetric case). This is also the case for LH models: the lightest extra states have masses in the TeV range (the scale at which the global symmetry is spontaneously broken), but their contributions to the electroweak observables are calculable and (hopefully) under control. Besides, this effective description is valid up to a cut-off scale, $\Lambda \simeq 10\,\mathrm{TeV}$, beyond which some unspecified UV completion takes over [4, 5].

Despite the good prospects, the absence of fine-tuning in particular LH scenarios should be checked in practice. More precisely, the fine-tuning must be computed for the different LH models with the same level of rigor employed for the supersymmetric models in the past. A systematic attempt of this kind has not been done up to now, and it is the main goal of this paper. We will focus only on the naturalness of the electroweak breaking, although LH models may have other (model-dependent) problems.

To quantify the fine tuning we follow Barbieri and Giudice [6, 7, 8]: we write the Higgs VEV as $v^2 = v^2(p_1, p_2, \ldots)$, where p_i are input parameters of the model under study, and define Δ_{p_i}, the amount of fine tuning associated to p_i, by

$$\frac{\delta M_Z^2}{M_Z^2} = \frac{\delta v^2}{v^2} = \Delta_{p_i} \frac{\delta p_i}{p_i}\,, \tag{1.4}$$

where δM_Z^2 (or δv^2) is the change induced in M_Z^2 (or v^2) by a change δp_i in p_i. Roughly speaking, $|\Delta_{p_i}^{-1}|$ measures the probability of a cancellation among terms of a given size

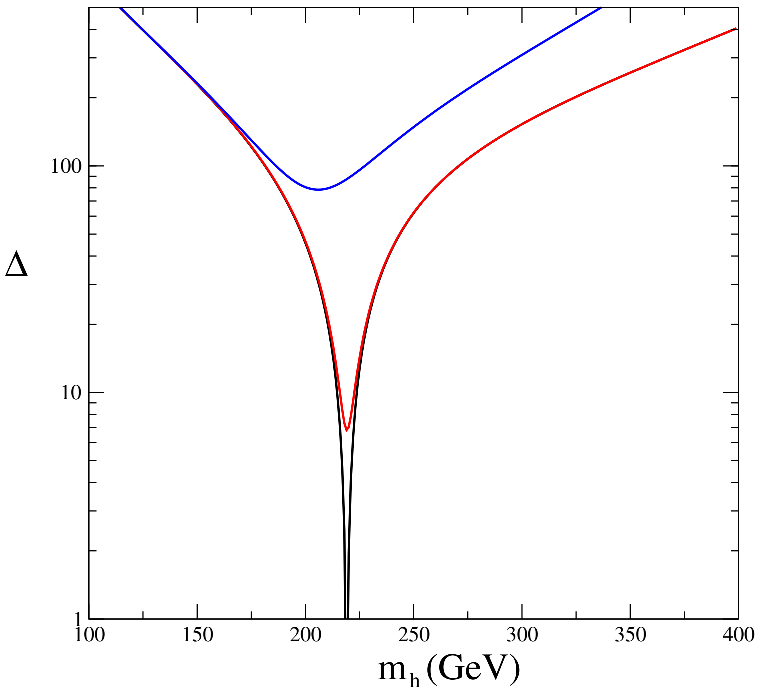

Figure 1: Fine-tuning contours as a function of the Higgs mass in the SM with a common cut-off $\Lambda = 10\,\text{TeV}$. This can be considered as the fine-tuning of the Little Hierarchy problem in the SM. Different curves correspond to progressively more sophisticated definitions of Δ (from black [bottom line] to red to blue [top line], see text for details).

to obtain a result which is $|\Delta_{p_i}|$ times smaller. Often, the fine-tuning in a given model is defined as $\Delta \equiv \text{Max}\{\Delta_{p_i}\}$. As discussed in appendix A, we prefer to define the total fine-tuning as

$$\Delta \equiv \left[\sum_i \Delta_{p_i}^2\right]^{1/2} . \tag{1.5}$$

In practice, however, we usually find that one of the Δ_{p_i} dominates over the rest and then, both definitions give similar fine-tunings. When several Δ_{p_i} are large, our definition gives larger fine-tuning (which makes sense), although the difference will never be dramatic. It is important to recall that the Little Hierarchy problem of the SM, which the LH models attempt to solve, is itself a fine-tuning problem: one could simply assume $\Lambda_{\text{SM}} \gtrsim 10\,\text{TeV}$ with the 'only' price of tuning $\delta_q m^2$, as given by eq. (1.2), at the 0.4–1 % level (or, equivalently, $\Delta = 100 - 250$). Therefore, to be of interest, the LH models should at least improve this degree of fine-tuning. In order to perform a fair comparison, this estimate can be refined following the lines explained in refs. [1, 9, 10]. First of all, eq. (1.2) should be renormalization-group improved. Then, the value of Δ ($= \Delta_\Lambda$) vs. m_h is given by the (bottom) black line of figure 1. The deep throat at $m_h \sim 220\,\text{GeV}$ results from an accidental cancellation between the various terms in eq. (1.2). This throat is cut when the fine-tuning parameter associated to the top mass (Δ_{λ_t}) is added in quadrature as explained above (for details see ref. [1]), giving the (middle) red line. Finally, once the fine-tuning parameter associated to the Higgs mass itself (Δ_λ) is included as well, the value of Δ is given by the (top) blue line, which thus represents the fine-tuning associated to the Little Hierarchy problem. This has to be compared with the tuning of LH models. On the other hand, for the minimal supersymmetric standard model (MSSM) the degree of fine-tuning is presently at the few percent level ($\Delta \simeq 20-40$ for $m_h \lesssim 125\,\mathrm{GeV}$), while for other supersymmetric models the situation is much better [11, 12]. Hence, in order to be competitive with supersymmetry, LH models should not worsen the MSSM performance. We will use these criteria in order to analyze the success of several representative LH models.

Due to the great variety of LH models we do not attempt to perform here an exhaustive analysis of them. Rather, we have focused on four LH scenarios [13]–[16] which are probably the most popular ones, and tried to extract general lessons for other models. The prototype LH scenario is the so-called Littlest Higgs model [13]. This model is a very good example to start with, due to its simplicity and because it shares many features with more elaborate LH constructions. Actually, many of those models are simply modifications of the Littlest Higgs model. The Littlest Higgs has some phenomenological problems with the constraints from precision electroweak observables. (Incidentally, this illustrates the fact that the impact of the TeV-mass states of LH models on electroweak observables is not always under control [17].) Since our focus is the naturalness of electroweak breaking, we will ignore those constraints, although the strongest results would come from combining both analyses. On the other hand, there exist modifications of the Littlest Higgs (also studied in this paper) able to overcome those difficulties.

The paper is organized as follows. In section 2 we analyze the structure, and evaluate the fine-tuning, of the Littlest Higgs model. Sections 3 and 4 are devoted respectively to the computation of the fine-tuning in two popular modifications of the Littlest Higgs proposed in refs. [14] and [15]. The latter corresponds to the so-called Littlest Higgs model with T-parity. In section 5 we study a recent proposal (the so-called "Simplest Little Higgs" [16]), whose structure differs substantially from the Littlest Higgs. In all these cases the fine-tuning turns out to be essentially comparable with that of the Little Hierarchy problem of the SM (that LH models attempt to solve) and higher than in supersymmetric models, and we discuss the reasons for this fact. Finally, in section 6 we summarize our results and present some conclusions. In addition we present in appendix A a simple recipe to evaluate the fine-tuning when the various parameters of a model are subject to constraints. Appendix B contains details on the structure of the different Little Higgs models studied.

2. The Littlest Higgs [13]

2.1 Structure of the model

The Littlest Higgs model is a non-linear sigma model based on a global SU(5) symmetry, spontaneously broken to SO(5) at a scale $f \sim 1\,\mathrm{TeV}$, and explicitly broken by the gauging of an $[\mathrm{SU}(2) \times \mathrm{U}(1)]^2$ subgroup. After the spontaneous breaking, the latter gets broken to its diagonal subgroup, identified with the SM electroweak gauge group, $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. From the 14 (pseudo)-Goldstone bosons of the SU(5) $\rightarrow$ SO(5) breaking, 4 degrees of freedom (d.o.f.) are true Goldstones [eaten by the gauge bosons of the spontaneously broken ("axial") SU(2) $\times$ U(1), which thus acquire masses $\sim f \sim 1\,\mathrm{TeV}$ through the Higgs

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mechanism] and the remaining 10 d.o.f. correspond to the SM Higgs doublet, $H = (h^0, h^+)$, (4 d.o.f.) and a complex $SU(2)_L$ scalar triplet, ϕ (6 d.o.f.) with $Y = 1$; in vectorial notation, $\phi = (\phi^{++}, \phi^+, \phi^0)$. All these fields can be treated simultaneously in a nonlinear matrix field Σ (see appendix B for details).

The $[SU(2) \times U(1)]^2$ gauge interactions give a radiative mass to the SM Higgs, but only when the couplings of both groups are simultaneously present, as explained in more detail below. Hence, the quadratically divergent contributions only appear at two-loop order, and the high-energy cut-off can be pushed up to a scale $\Lambda \sim 4\pi f \sim 10\,\text{TeV}$, as explained in the introduction. For the potentially dangerous top-Yukawa interactions things work in a similar way: the spectrum is enlarged with two extra fermions of opposite chiralities, and the conventional top-Yukawa coupling, λ_t, is not an input parameter but results from two independent couplings λ_1, λ_2. Both must be present in order to generate a radiative correction to the Higgs mass, and this again forbids quadratically divergent corrections to m_h^2 at one-loop.

For our purposes, the relevant states besides those of the SM are: the pseudo-Goldstone bosons H, ϕ; the heavy gauge bosons, W', B', of the axial $SU(2) \times U(1)$; and the two extra (left and right) fermionic d.o.f. that combine in a vector-like "heavy Top", T. The relevant part of the lagrangian can be found in appendix B, eqs. (B.8) and (B.10). It consists of two pieces

$$\mathcal{L} = \mathcal{L}_{\text{kin}}(g_1, g_2, g_1', g_2') + \mathcal{L}_f(\lambda_1, \lambda_2)\,, \tag{2.1}$$

where g_1, g_1' (g_2, g_2') are the gauge couplings of the first (second) $SU(2) \times U(1)$ factor, and λ_1, λ_2 are the two independent fermionic couplings. These couplings are constrained by the relations with the SM couplings,

$$\frac{1}{g^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}\,, \qquad \frac{1}{g'^2} = \frac{1}{g_1'^2} + \frac{1}{g_2'^2}\,, \qquad \frac{2}{\lambda_t^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\,, \tag{2.2}$$

where g and g' are the $SU(2)$ and $U(1)_Y$ gauge couplings, respectively, and λ_t is the top Yukawa coupling. The lagrangian (2.1) gives $\mathcal{O}(f)$ masses to W', B' and T. These heavy masses have a non-trivial dependence on the full non-linear field Σ, which contains the H and ϕ fields. In particular, retaining only the dependence on $h \simeq \text{Re}(h^0)\sqrt{2}$ we get

$$\begin{aligned}
m_{W'}^2(h) &= M_{W'}^2 + \mathcal{O}(h^2) = \frac{1}{4}(g_1^2 + g_2^2)f^2 - \frac{1}{4}g^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\
m_{B'}^2(h) &= M_{B'}^2 + \mathcal{O}(h^2) = \frac{1}{20}(g_1'^2 + g_2'^2)f^2 - \frac{1}{4}g'^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\
m_T^2(h) &= M_T^2 + \mathcal{O}(h^2) = (\lambda_1^2 + \lambda_2^2)f^2 - \frac{1}{2}\lambda_t^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right).
\end{aligned} \tag{2.3}$$

At this level, H and ϕ are massless, but they get massive radiatively. The simplest way to see this is by using the effective potential. Let us consider first the quadratically divergent contribution to the one-loop scalar potential, given by

$$V_1^{\text{quad}} = \frac{1}{32\pi^2}\Lambda^2\, \text{Str}\mathcal{M}^2\,, \tag{2.4}$$

where the supertrace Str counts degrees of freedom with a minus sign for fermions, and $\mathcal{M}^2$ is the (tree-level, field-dependent) mass-squared matrix. In our case, the previous formula gives

$$V_1^{\rm quad} = \frac{1}{32\pi^2}\Lambda^2 \, \left[6m_W^2 + 9m_{W'}^2 + 3m_Z^2 + 3m_{B'}^2 - 12(m_t^2 + m_T^2)\right] . \tag{2.5}$$

By looking at the h-dependence of the masses above it is easy to check that $V_1^{\rm quad}$ does not contain a mass term for h (this will be generated by the logarithmic and finite contributions to the potential, to be discussed shortly). The reason for this result is the following. If $\lambda_1 = g_2 = g_2' = 0$, the lagrangian (2.1) recovers a global SU(3) [SU(3)$_1$, living in the upper corner of SU(5)] that protects the mass of the Higgs (which transforms by a shift under that symmetry). On the other hand, if $\lambda_2 = g_1 = g_1' = 0$, then a different SU(3) symmetry [SU(3)$_2$, living in the lower corner of SU(5)] is recovered that also protects the Higgs mass. A non-zero value for the Higgs mass can only be generated by breaking both SU(3)'s and therefore both type-1 and type-2 couplings should be present. Quadratically divergent diagrams involve only one type of coupling and therefore cannot contribute to the Higgs mass. This is the so-called collective breaking of the original SU(5) symmetry and is one of the main ingredients of Little Higgs models.

These symmetries do not protect the mass of the triplet. In fact, if we include the full dependence of the bosonic (W', B') and fermionic (T) masses on the Σ field, $V_1^{\rm quad}$ contains operators, $\mathcal{O}_V(\Sigma)$ and $\mathcal{O}_F(\Sigma)$ respectively, that produce a mass term for the triplet ϕ of order $\Lambda^2/(16\pi^2) \sim f^2$. Explicit expressions for these operators are given in appendix B. Then, following [13], it is reasonable to assume that $\mathcal{O}_V(\Sigma)$ and $\mathcal{O}_F(\Sigma)$ are already present at tree-level, as a remnant of the heavy physics integrated out at Λ (a threshold effect). These effects can be accounted for by adding an extra piece to the lagrangian,

$$-\,\Delta\mathcal{L} = c\,\mathcal{O}_V(\Sigma) + c'\,\mathcal{O}_F(\Sigma)\,, \tag{2.6}$$

were c and c' are unknown coefficients [see eq. (B.11) in appendix B for an explicit expression of $\Delta\mathcal{L}$]. For future use, it is convenient to discuss here what is the natural size of c and c'. Naive dimensional analysis [18] has been used to estimate $c, c' \sim \mathcal{O}(1)$. We can make a more precise evaluation by computing the one-loop contributions to c and c' coming from (2.5), keeping the full dependence of the masses on Σ. Then we get

$$\begin{aligned} c &= c_0 + c_1 = c_0 + \frac{3}{4}\,, \\ c' &= c_0' + c_1' = c_0' - 24\,. \end{aligned} \tag{2.7}$$

where the subindex 0 labels the unknown threshold contributions from physics beyond Λ.

Besides giving a mass to ϕ, the operators in eq. (2.6) produce a coupling $\sim h^2\phi$[1] and a quartic coupling for h. This quartic coupling is modified by the presence of the $h^2\phi$ term

[1]This coupling induces a tadpole for ϕ after electroweak symmetry breaking. Keeping the VEV of ϕ small enough is a necessary requirement to obtain an acceptable model and we ensure that this is the case in our numerical analysis. Then, it is a good approximation to neglect the effect of that small VEV in most places.

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once the heavy triplet is integrated out. After that is done, the Higgs quartic coupling λ can be written in the simplest manner as

$$\frac{1}{\lambda} = \frac{1}{\lambda_a} + \frac{1}{\lambda_b}\,, \tag{2.8}$$

with

$$\lambda_a \equiv c(g_2^2 + g_2'^2) - c'\lambda_1^2\,, \qquad \lambda_b \equiv c(g_1^2 + g_1'^2)\,. \tag{2.9}$$

We see that the structure of (2.8) is similar to that of (2.2) for the fermion and gauge boson couplings, with λ_a (λ_b) being a type-1 (type-2) coupling.

In order to write the one-loop Higgs potential, we need explicit expressions for the h-dependent masses of the spectrum. In the scalar sector, we decompose $h^0 \equiv (h^{0r} + ih^{0i})/\sqrt{2}$ and $\phi^0 \equiv i(\phi^{0r} + i\phi^{0i})/\sqrt{2}$. In the CP-even sector we write the relevant part of the mass matrix in the basis $\{h^{0r}, \phi^{0r}\}$; in the CP-odd sector we use the basis $\{h^{0i}, \phi^{0i}\}$ and finally, in the charged sector the basis $\{h^+, \phi^+\}$. The three mass matrices are very similar in structure and can be written simultaneously as[2]

$$M_\kappa^2(h) = \begin{bmatrix} \frac{1}{4}a_\kappa\lambda_+ h^2 + \frac{1}{\sqrt{2}}s_\kappa\lambda_- ft + \mathcal{O}(h^4/f^2) & b_\kappa\lambda_- fh + \mathcal{O}(h^2) \\ & \\ b_\kappa^*\lambda_- fh + \mathcal{O}(h^2) & \lambda_+(f^2 - c_\kappa h^2) + \mathcal{O}(h^4/f^2) \end{bmatrix}, \tag{2.10}$$

where the index $\kappa = \{0r, 0i, +\}$ labels the different sectors, $a_\kappa = \{3,1,1\}$, $s_\kappa = \{1,-1,0\}$, $b_\kappa = \{1/\sqrt{2}, 1/\sqrt{2}, i/2\}$, $c_\kappa = |b_\kappa|^2$, and we have defined $\lambda_+ \equiv \lambda_a + \lambda_b$, $\lambda_- \equiv \lambda_a - \lambda_b$. We have also included in these mass matrices the contribution of the triplet VEV, $t \equiv \langle\phi^{0r}\rangle$, with

$$t \simeq -\frac{1}{2\sqrt{2}}\frac{\lambda_- h^2}{\lambda_+ f}\,. \tag{2.11}$$

The off-diagonal entries in (2.10) are due to the $h^2\phi$ coupling and they cause mixing between h and ϕ. Concerning the masses, the effect of this mixing is negligible for the triplet [at order h^2, the masses of ϕ^{0r} and ϕ^{0i} are the same, and these fields can still be combined in a complex field ϕ^0]. Explicitly, these masses are

$$\begin{bmatrix} m_{\phi^0}^2(h) \\ m_{\phi^+}^2(h) \\ m_{\phi^{++}}^2(h) \end{bmatrix} = M_\phi^2 + \mathcal{O}(h^2) = (\lambda_a + \lambda_b)f^2 - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\lambda h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \tag{2.12}$$

We will call h'^{0r}, h'^{0i} and h'^+ the light mass eigenstates of (2.10) in the different sectors, for which we get

$$\begin{bmatrix} m_{h'^{0r}}^2(h) \\ m_{h'^{0i}}^2(h) \\ m_{h'^+}^2(h) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\lambda h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \tag{2.13}$$

[2] At this point there is no tree-level mass term for the Higgs field but the presence of a quartic coupling gives it a nonzero mass in a background h.

From the previous expressions it is straightforward to check that, in the contribution of scalars to $V_1^{\rm quad}$,

$$\frac{\Lambda^2}{32\pi^2}\left(m^2_{h'^{0r}}+m^2_{h'^{0i}}+2m^2_{h'^+}+2m^2_{\phi^0}+2m^2_{\phi^+}+2m^2_{\phi^{++}}\right), \tag{2.14}$$

there is also a cancellation of h^2 terms. This is due to the fact that the operators of (2.6) still respect the same $SU(3)_i$ symmetries of the original lagrangian as they originate from quadratically divergent one-loop corrections.

Finally, a non-vanishing mass parameter for h arises from the logarithmic and finite contributions to the effective potential. In the $\overline{\rm MS}$ scheme, in Landau gauge, and setting the renormalization scale $Q = \Lambda$,

$$\begin{aligned} m^2 = \frac{3}{64\pi^2}&\left\{3g^2 M^2_{W'}\left[\log\frac{\Lambda^2}{M^2_{W'}}+\frac{1}{3}\right]+g'^2M^2_{B'}\left[\log\frac{\Lambda^2}{M^2_{B'}}+\frac{1}{3}\right]\right\}+\\ &+\frac{3\lambda}{8\pi^2}M^2_\phi\left[\log\frac{\Lambda^2}{M^2_\phi}+1\right]-\frac{3\lambda_t^2}{8\pi^2}M^2_T\left[\log\frac{\Lambda^2}{M^2_T}+1\right], \end{aligned} \tag{2.15}$$

where we have included the contribution from the ϕ masses.

In summary, the effective potential of the Higgs field can be written in the SM-like form

$$V=\frac{1}{2}m^2h^2+\frac{1}{4}\lambda h^4\,, \tag{2.16}$$

where λ and m^2 are given by eqs. (2.8) and (2.15). The Higgs VEV is simply

$$v^2=-\frac{m^2}{\lambda}\,. \tag{2.17}$$

2.2 Fine-tuning analysis

A rough estimate of the fine-tuning associated to electroweak breaking in the Littlest Higgs model can be obtained from eq. (2.15). The contribution of the heavy Top, T, to the Higgs mass parameter is

$$\delta_T m^2=-\frac{3\lambda_t^2}{8\pi^2}M_T^2\left[\log\frac{\Lambda^2}{M_T^2}+1\right]. \tag{2.18}$$

Using eqs. (2.2) and (2.3), it follows[3] that $M_T^2 \geq 2\lambda_t^2 f^2$, and thus $\delta_T m^2 \geq 0.37 f^2$ (the minimum corresponds to $\lambda_1 = \lambda_2 = \lambda_t$). Thus the ratio $\delta_T m^2/m^2$, tends to be quite large: e.g. for $f = 1\,\text{TeV}$ and $m_h = 115, 150, 250\,\text{GeV}$, one gets $|\delta_T m^2/m^2| \geq 56,\ 33,\ 12$ respectively. Since there are other potential sources of fine-tuning, this should be considered as a lower bound on the total fine-tuning. Actually, the overall fine tuning is usually much larger than this estimate, as we show below. (Eventually we will go back to this rough argument to improve it in a simple way.)

In order to perform a complete fine-tuning analysis we determine first the input parameters, p_i, and then calculate the associated fine-tunings, Δ_{p_i}, according to eq. (1.4),

[3]Similar bounds, based on the same type of coupling structure, hold for the rest of heavy states: $M^2_{W'} \geq g^2 f^2$, $M^2_{B'} \geq g'^2 f^2/5$ and $M^2_\phi \geq 4\lambda f^2$.

i.e. $\Delta_{p_i} = (p_i/v^2)(\partial v^2/\partial p_i)$. For the Littlest Higgs model the input parameters of the lagrangian [eqs. (2.1) and (2.6)] are

$$p_i = \{g_1,\ g_2,\ g_1',\ g_2',\ \lambda_1,\ \lambda_2,\ c,\ c',\ f\}\,. \tag{2.19}$$

We have not included Λ among these parameters since we are assuming $\Lambda \simeq 4\pi f$. On the other hand, the parameter f basically appears as a multiplicative factor in the mass parameter, m^2, so Δ_f is always $\mathcal{O}(1)$, and can be ignored.[4] Finally, the above parameters are constrained by the measured values of the top mass and the gauge couplings g, g', according to eq. (2.2). The procedure to estimate the fine-tuning in the presence of constraints is discussed in appendix A. The net effect is a reduction of the "unconstrained" total fine-tuning, $\Delta = (\sum_i \Delta_{p_i}^2)^{1/2}$, according to eq. (A.7). In this particular case, that equation gives

$$\Delta = \left[\sum_i \Delta_{p_i}^2 - \sum_{\alpha=1}^{3} \frac{1}{N_\alpha^2}\left(\sum_i p_i \frac{\partial G_\alpha^{(0)}}{\partial p_i}\Delta_{p_i}\right)^2\right]^{1/2}, \tag{2.20}$$

where $G_\alpha^{(0)} = \{g^2, g'^2, \lambda_t^2\}$ are functions of the p_i as given in eq. (2.2), and

$$N_\alpha^2 \equiv \sum_i p_i^2 \left(\frac{\partial G_\alpha^{(0)}}{\partial p_i}\right)^2, \tag{2.21}$$

are normalization constants.

As announced before, Δ is in general much larger than the initial rough estimate, although the precise magnitude depends strongly on the region of parameter space considered and decreases significantly as m_h increases. Let us discuss how this comes about. The negative contribution from M_T^2 to m^2 in eq. (2.15) must be compensated by other positive contributions. Typically, this requires a large value of the triplet mass, $M_\phi^2 = (\lambda_a + \lambda_b)f^2$, which requires a large value of $(\lambda_a + \lambda_b)$, but keeping $1/\lambda = 1/\lambda_a + 1/\lambda_b$ fixed for a given m_h. There are two ways of achieving this:[5]

$$\begin{aligned} &\text{a)}\ \lambda \simeq \lambda_b \ll \lambda_a \simeq \frac{M_\phi^2}{f^2}\,, \\ &\text{b)}\ \lambda \simeq \lambda_a \ll \lambda_b \simeq \frac{M_\phi^2}{f^2}\,. \end{aligned} \tag{2.22}$$

Notice that the one-loop m^2 is a symmetric function of λ_a and λ_b, so cases a) and b) are simply related by $\lambda_a \leftrightarrow \lambda_b$. This means that the triplet and Higgs masses are exactly the same in both cases although the fine-tuning may be different (since the dependence of $\lambda_{a,b}$ on p_i is not the same), and indeed it is, as we discuss next.

[4] Now it is clear that the assumption $\Lambda \simeq 4\pi f$ reduces the amount of fine-tuning. Had we kept $\{\Lambda, f\}$ as input parameters, variations of Λ or f would have produced large changes in m^2, and thus in v^2. Therefore, this assumption is a conservative one.

[5] The existence of two separate regions of solutions can be also understood from the fact that the minimization condition (2.17) becomes quadratic in c, for given values of $\lambda, \lambda_1, g_1, g_1'$, and in the approximation $\log(\Lambda^2/M_\phi^2) \simeq h$-independent.

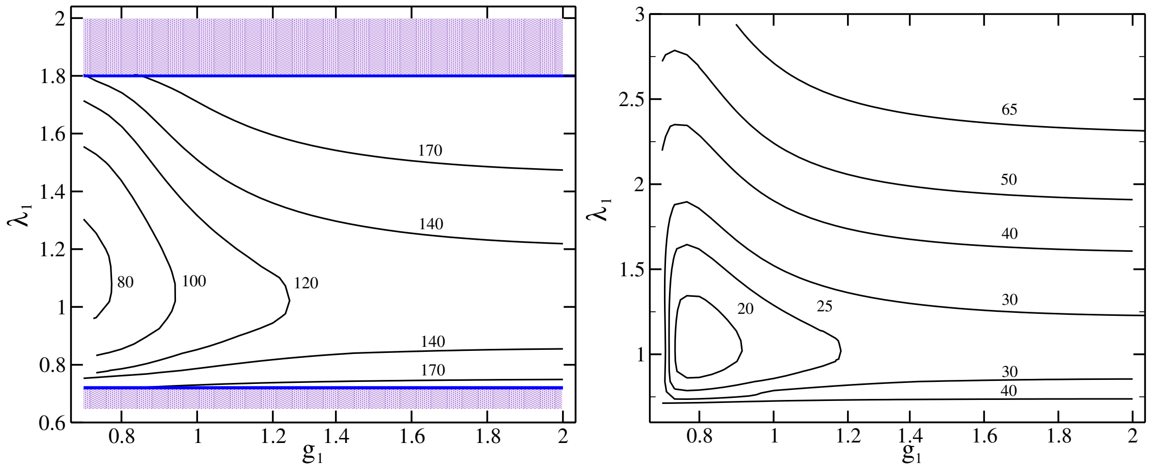

Figure 2: Preliminary fine-tuning contours for the Littlest Higgs model, case a) of eq. (2.22), for two different values of the Higgs mass: $m_h = 115\,\mathrm{GeV}$ (left) and $m_h = 250\,\mathrm{GeV}$ (right).

For case a), the value of Δ is shown by the contour plots of figure 2 which correspond to two different values of the Higgs mass. We present our results in the plane $\{g_1, \lambda_1\}$. In each point of this plane, g_2 and λ_2 are then fixed by eq. (2.2); the values of c and c' are fixed by the minimization condition for electroweak breaking and the choice of Higgs mass. The value of g'_1 has been taken at $g'^2_1 = g'^2_2 = g'^2/2$, which nearly minimizes the fine-tuning. (Note also that $g_1 \geq g$ and thus smaller values of Δ cannot be reached by lowering g_1 in figure 2.) The shaded areas correspond to regions that do not give a correct electroweak symmetry breaking (in these regions, $M_\phi \geq \Lambda$, which besides being beyond the range of validity of the effective theory, makes negative the triplet contribution to m^2). These plots illustrate the large size of Δ, which is significantly larger than the previous rough estimate. This is not surprising since, as stated before, besides the heavy top contribution to m^2 (on which the estimate was based), there are other contributions that depend in various ways on the different input parameters. This gives additional contributions to the total fine-tuning, increasing its value. The plots also show how Δ decreases for increasing m_h. This is due to the fact that the larger m_h, and thus λ, the larger the required value of m^2 in (2.17), which reduces the level of cancellation needed between the various contributions to m^2 in (2.15) [1]. Although the fine-tuning is substantial, it could be considered as tolerable [i.e. $\mathcal{O}(10)$], for some (small) regions of parameter space, at least for large m_h. However, on closer examination the fine-tuning turns out to be larger than shown by figure 2. From the condition a) in (2.22)

$$\lambda \simeq c(g_1^2 + g'^2_1) = \lambda_b \ll \lambda_a = c(g_2^2 + g'^2_2) - c'\lambda_1^2\,, \tag{2.23}$$

it is clear that in this case c' is large (and negative), while c is small. But then, eq. (2.7) shows that there is an implicit tuning between c_0 and c_1 to get the small value of c. In fact, it makes more sense to include c_0 and c'_0, rather than c and c', among the unknown input parameters appearing in (2.19). Then, since $\Delta_{c_0} = |(c_0/c)\Delta_c|$ (and similarly for $\Delta_{c'_0}$), the global fine-tuning becomes much larger. This is illustrated in figure 3 (upper plots), where Δ is systematically above $\mathcal{O}(10)$, even for large m_h.

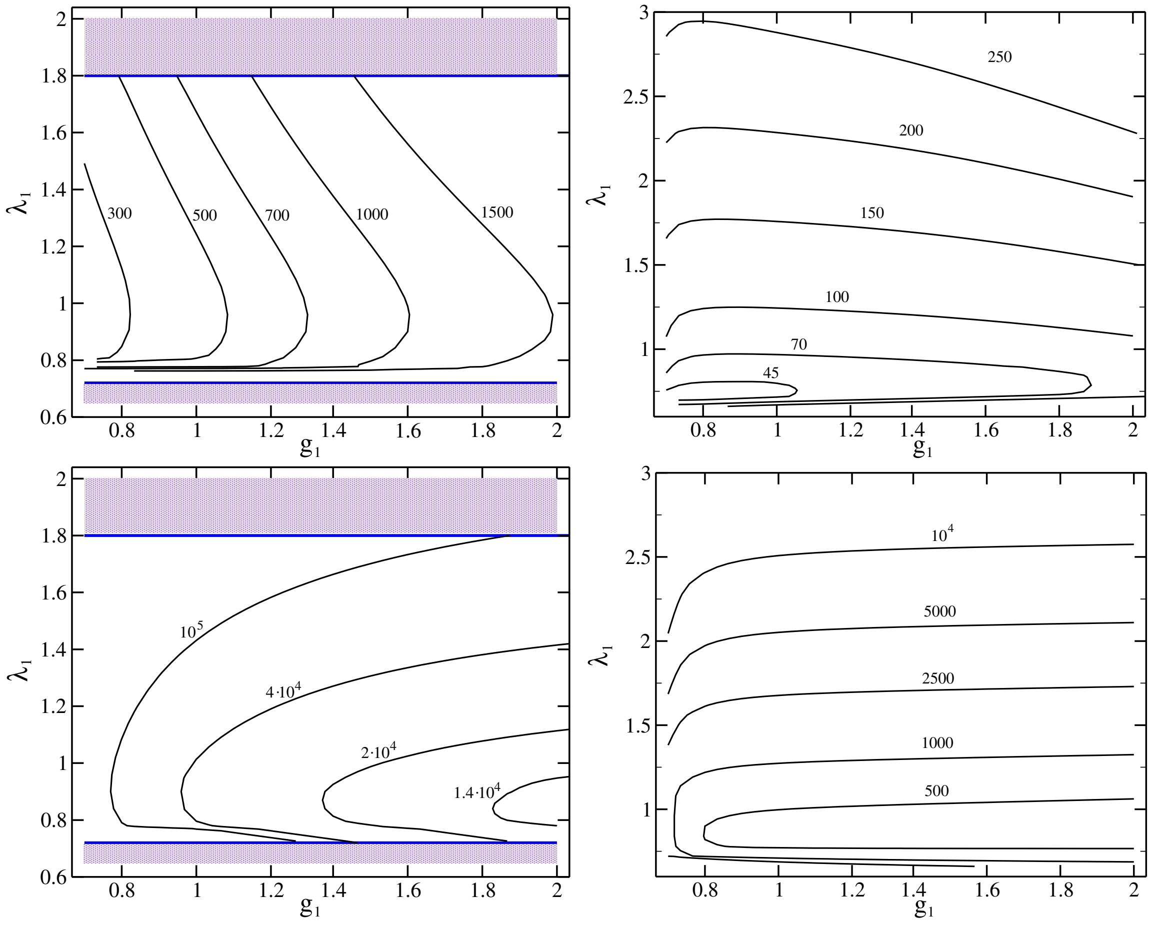

Figure 3: Final fine-tuning contours for the Littlest Higgs model, using c_0 and c_0' of eq. (2.7) as unknown parameters to improve the analysis. TOP: Case a) of eq. (2.22) for $m_h = 115\,\mathrm{GeV}$ (left) and $m_h = 250\,\mathrm{GeV}$ (right). BOTTOM: The same but for case b) of eq. (2.22).

There is a simple way of understanding the order of magnitude of Δ. We can repeat the rough argument at the beginning of this subsection, but considering now the contribution of the triplet to the Higgs mass parameter in (2.15). More precisely, since $M_\phi^2 = (\lambda_a + \lambda_b) f^2 = [c(g_1^2 + g_1'^2 + g_2^2 + g_2'^2) - c'\lambda_1^2] f^2$, we can focus on the contribution proportional to c':

$$\delta_{c'} m^2 = -\frac{3\lambda}{8\pi^2} c' \lambda_1^2 f^2 \left[\log \frac{\Lambda^2}{M_\phi^2} + 1\right]. \tag{2.24}$$

Now, c' itself contains a radiative piece $c_1' = -24$ [see eq. (2.7)], whose relative contribution to m^2 is then given by

$$\left|\frac{\delta_{c_1'} m^2}{m^2}\right| \geq \frac{9}{2\pi^2} \lambda_t^2 \frac{f^2}{v^2} \left[\log \frac{\Lambda^2}{M_\phi^2} + 1\right] \simeq 45\,, \tag{2.25}$$

where we have first used $\lambda_1^2 \geq \lambda_t^2/2$ and then $M_\phi \sim f$. Hence we easily expect $\mathcal{O}(100)$ contributions to Δ, as reflected in figure 3.

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It is interesting to note that this rough argument holds even if there are additional contributions to m^2, since it is based on the size of contributions that are present anyway. In particular, two-loop corrections or 'tree-level' (i.e. threshold) corrections to m^2 are not likely to help in improving the fine-tuning. Of course, it might happen that they have just the right size to cancel the known large contributions, such as those of eqs. (2.18) and (2.25). However, in the absence of a theoretical argument for that cancellation, this possibility can only be understood a priori as a fortunate accident. The chances for the latter are precisely what the fine-tuning analysis evaluates.

For case b) in eq. (2.22) things are much worse, as illustrated in figure 3 (lower plots), which shows huge values of Δ. The reason is the following. In case b), both c and c' are sizeable, so there is no implicit tuning between c_0 (c'_0) and c_1 (c'_1), but this implies a cancellation to get $\lambda_a = c(g_2^2 + g_2'^2) - c'\lambda_1^2 \simeq \lambda$, which requires a delicate tuning. This "hidden fine-tuning" is responsible for the unexpectedly large values of Δ. In other words, small changes in the input parameters of the model produce large changes in the value of λ, and thus in the value of v^2.

Now, imagine some future time after the Higgs mass has already been measured so that the parameter λ takes a particular value and the other parameters of the model can only be varied in such a way that λ remains constant. (As the fine-tuning is a kind of statistical guess, it gets modified when the experimental knowledge varies, like any conditional probability [8]. Fine-tuning arguments are based on the observation that the large tuning of some unknown parameter is unlikely. On the other hand, a precisely tuned parameter that has already been measured can be taken as evidence for some theoretical correlation, or be simply due to a lucky accident, but assigning to it a probability can be misleading.) Then, according to the above discussion, the fine-tuning for case b) should be dramatically reduced and, apparently, this is exactly what happens. The condition of constant λ can be incorporated in the computation of Δ using eq. (A.7) with an additional constraint $G_4^{(0)} = \lambda$.[6] The new "constrained" fine tuning in case b) (for $m_h = 115\,\mathrm{GeV}$), is shown in the left plot of figure 4, to be compared with the bottom-left plot of figure 3. Although still sizeable, the fine-tuning is now much smaller.

However, this behaviour does not alleviate the fine-tuning problems. If the Higgs mass is measured, one can also consider what is the fine-tuning between the input parameters of the model to produce such value of m_h, in the same way that one examines the fine-tuning to produce the measured value of v^2. Let us denote the fine-tuning in m_h^2 (or equivalently in λ) associated to a parameter p_i by $\Delta_{p_i}^{(\lambda)}$. It is given by

$$\frac{\delta\lambda}{\lambda} = \Delta_{p_i}^{(\lambda)} \frac{\delta p_i}{p_i}\,. \tag{2.26}$$

The right plot in figure 4 shows that the values of $\Delta^{(\lambda)}$ are quite large, as expected. If $\Delta^{(\lambda)} > \mathcal{O}(1)$, this fine-tuning must be taken into account and, since Δ and $\Delta^{(\lambda)}$ represent independent inverse probabilities, they should be multiplied to estimate the total fine-

[6]The constraint $G_4^{(0)} = \lambda$ is not independent of the others (for g^2, g'^2 and λ_t). A Gramm-Schmidt orthonormalization of the different constraints is enough to deal with this complication (see appendix A).

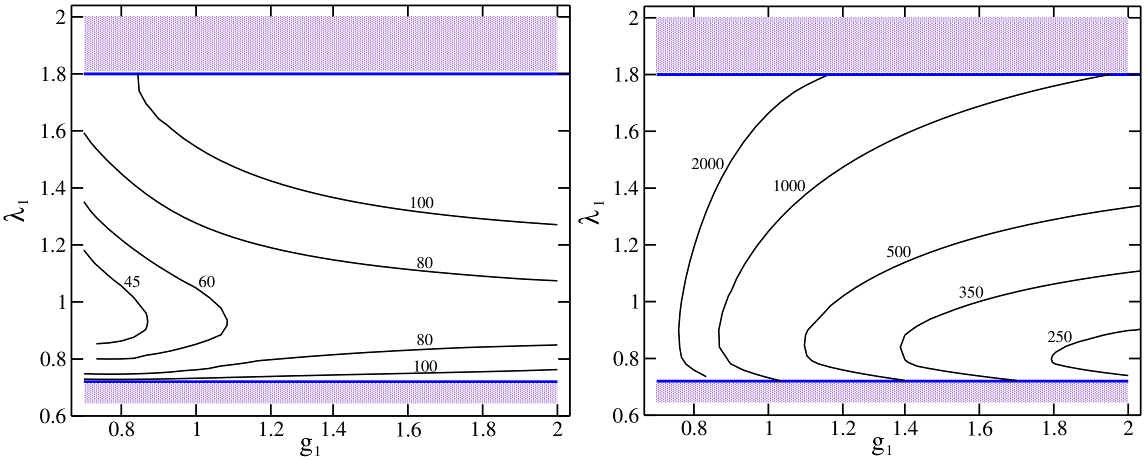

Figure 4: LEFT: The same as the bottom-left plot of figure 3, but keeping λ fixed. RIGHT: Fine-tuning associated to λ itself in the same case.

tuning $\Delta \cdot \Delta^{(\lambda)}$ in the model. This fine-tuning turns out to be very large, comparable to the values of Δ before the measurement of m_h.

The final conclusion is that the "standard" Littlest Higgs model has built-in a significant fine-tuning problem, especially for $m_h < 250\,\text{GeV}$, even if other problems with electroweak observables are ignored. In this range the fine-tuning is typically $\Delta \gtrsim \mathcal{O}(100)$, i.e. essentially of the same order (or higher) than that of the Little Hierarchy problem of the SM [see figure 1] and more severe than the MSSM one. For larger values of m_h, which is not so attractive from the point of view of fits to electroweak observables[7] [20], the situation is better, although still $\Delta > 10$. The final results of this section are summarized by figure 3.

Let us finish this subsection with two additional comments. First, notice that the plots presented correspond to $f = 1\,\text{TeV}$, which is a desirable and standard value in Little Higgs models. For other values of f, the parametric dependence of the fine-tuning is roughly $\Delta \propto f^2$ (there are also logarithmic effects that scale differently, and depend on whether one insists in keeping $\Lambda = 10\,\text{TeV}$ or $\Lambda = 4\pi f$). In fact, precision electroweak observables in the Littlest Higgs model require larger values of the masses of the new particles and therefore of f [17], which makes the fine-tuning even more severe. The second comment concerns perturbativity. We have just seen that a large value of c' [and also c for region b) in eq. (2.22)] is generically required for a correct electroweak breaking. Actually, from eq. (2.7), it seems indeed natural to expect large values of c', which might be a problem for perturbativity. One way of obtaining a smaller value of c' would be to lower Λ, making it smaller than $4\pi f$, which reduces the low-energy radiative contribution to c'. In fact it is well known [21] that chiral perturbation theory as a low energy description of technicolor theories with a large number of technifermions, N, breaks down at the scale $4\pi f/\sqrt{N}$. In the Littlest Higgs model we do have a large number of degrees of freedom (*e.g.* 12 only

[7]In models with a triplet VEV there can be cancellations that allow for larger Higgs masses giving a good fit. This can happen in the Littest Higgs model, see [19], although the required value of f is quite high.

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from T) so, the low-energy effective theory would not be reliable all the way up to $4\pi f$. (A similar result follows from considering unitarity limits [22].) Conversely, if one insists in keeping $\Lambda \simeq 4\pi f/\sqrt{N} \simeq 10\,\mathrm{TeV}$ to solve the Little Hierarchy problem, one would need f larger than 1 TeV. This would help with the fits to precision electroweak measurements but would worsen significantly the fine-tuning.

3. A modified version of the Littlest Higgs model [14]

This model [14] is very similar to the Littlest Higgs, except for the fact that the gauged subgroup of SU(5) is $[\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y]$, rather than $[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$. The absence of the heavy B' gauge boson helps with precision electroweak fits [14, 23], which is the main motivation for this model. The price to pay for not doubling the gauged U(1) is that the Higgs mass is not protected from quadratically divergent radiative corrections involving $\mathrm{U}(1)_Y$ interactions even at one-loop level. However, those corrections are not especially dangerous, due to the smallness of the g' coupling. Otherwise, the structure of the model is very similar to the Littlest Higgs [in particular, the lagrangian contains pieces similar to (2.1) and (2.6), see appendix B.2 for details]. The input parameters of the model are now

$$p_i = \{g_1,\ g_2,\ \lambda_1,\ \lambda_2,\ c,\ c',\ f\}\,, \tag{3.1}$$

to be compared with (2.19) for the Littlest Higgs model. As in that model, f can be ignored for the fine-tuning analysis.

For the fine-tuning analysis we need the h-dependent masses, which enter the one-loop effective potential. These are collected in appendix B.2. Besides the absence of g_1' and g_2', the main difference with the original Littlest Higgs model is that the Higgs mass parameter m^2 gets an additional positive contribution from the operator $c\ \mathcal{O}_V(\Sigma)$ (the form of this operator is dictated by the quadratically divergent contribution from gauge boson loops, see appendix B.2),

$$\delta m^2 = cg'^2\frac{\Lambda^2}{16\pi^2} = cg'^2 f^2\,. \tag{3.2}$$

This contribution involves g' as anticipated. Adding the one-loop logarithmic corrections we get

$$\begin{aligned} m^2 = {} & cg'^2 f^2 + \frac{9g^2}{64\pi^2}M_{W'}^2\left[\log\frac{\Lambda^2}{M_{W'}^2}+\frac{1}{3}\right] - \frac{3\lambda_t^2}{8\pi^2}M_T^2\left[\log\frac{\Lambda^2}{M_T^2}+1\right] + \\ & + \frac{3}{8\pi^2}\left\{\left(\lambda+\frac{17}{12}cg'^2\right)M_\phi^2\left[\log\frac{\Lambda^2}{M_\phi^2}+1\right] - \left(\lambda+\frac{1}{12}cg'^2\right)M_s^2\left[\log\frac{\Lambda^2}{M_s^2}+1\right]\right\}, \end{aligned} \tag{3.3}$$

where the Higgs quartic coupling is now

$$\lambda = \frac{1}{4}\left[\lambda_a' + \lambda_b' - \frac{4}{3}cg'^2 - \frac{(\lambda_a'-\lambda_b')^2}{(\lambda_a'+\lambda_b'+4cg'^2)}\right], \tag{3.4}$$

with $\lambda_a' \equiv cg_2^2 - c'\lambda_1^2$ and $\lambda_b' \equiv cg_1^2$. The expression for M_T is as for the Littlest Higgs, the triplet mass is $M_\phi^2 = (\lambda_a'+\lambda_b'+4cg'^2)f^2$ and $M_s^2 = cg'^2 f^2$ is the squared mass associated to

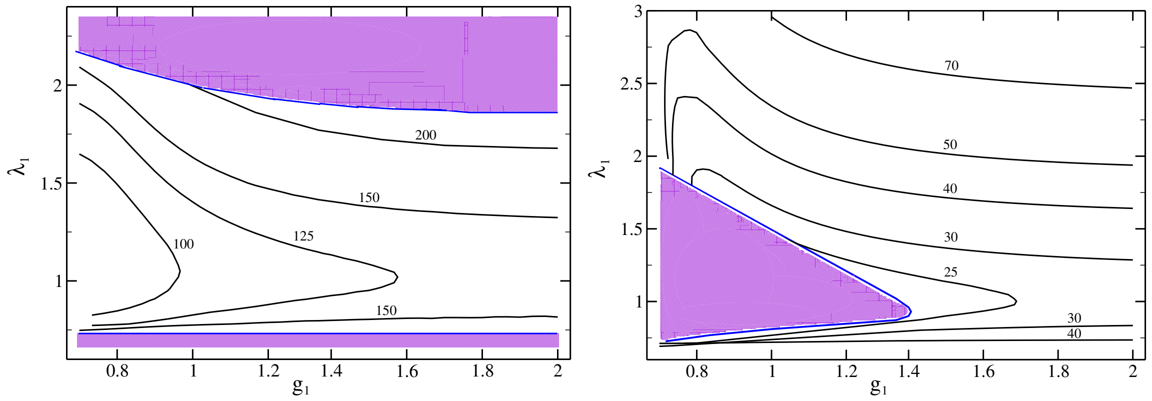

Figure 5: Preliminary fine-tuning contours, using c and c' as unknown parameters, for the Little Higgs model of [14], case a), with $m_h = 115\,\mathrm{GeV}$ (left plot) and $m_h = 250\,\mathrm{GeV}$ (right plot).

the light Higgses (see appendix B.2). Eqs. (3.3) and (3.4) have to be compared with (2.15) and (2.8) for the Littlest Higgs.

The presence of the g' terms in m^2 complicates the parameter dependence of the minimization condition for electroweak breaking: c and c' do no longer enter in m^2 just through λ'_a and λ'_b. Nevertheless, there are still two separate regions of solutions, which are the respective heirs of the two regions named a) and b) for the Littlest Higgs model [eq. (2.22)];[8] thus we keep the same notation.

The fine-tuning Δ for the region a), using c and c' as input parameters, is shown in figure 5. The magnitude of Δ is similar to that in the Littlest Higgs model, figure 2. In the present case the tree-level contribution cg'^2f^2 in (3.3), which is positive,[9] helps in compensating the negative correction from the heavy Top, so that the contribution from the triplet, and thus the triplet mass M_ϕ^2, is not required to be as large as before. Consequently, the values of c and c' will be smaller, as happened (for c) in the region a) of the Littlest Higgs model. However, as discussed in the previous section, small c and c' cause additional fine-tuning,[10] which can be taken into account by using c_0 and c'_0, rather than c and c' as the input parameters appearing in (3.1). This enhancement of the fine-tuning can be appreciated in the corresponding plots [both for a) and b) regions] in figure 6.

Figure 6 represents our final results for the model analyzed in this section. The fine-tuning is quite similar to that for the Littlest Higgs model, as summarized in figure 3. Therefore, the same comments apply here: the fine-tuning is always substantial ($\Delta > 10$) and for $m_h < 250\,\mathrm{GeV}$ is essentially of the same order as (or higher than) that of the Little Hierarchy problem [$\Delta \gtrsim \mathcal{O}(100)$] and worse than in the MSSM. As in the Littlest Higgs, two-loop or 'tree-level' contributions to m^2 are not likely to improve the situation [note in particular that eqs. (2.18) and (2.25) remain the same in this scenario].

[8] Again, the existence of these two regions can be understood here using the approximation explained in footnote 5.

[9] For $c < 0$ one breaks the electroweak symmetry at tree-level. However, this possibility leads to a large VEV for the triplet and therefore we focus on $c > 0$.

[10] Note that eq. (2.7) holds also in this model.

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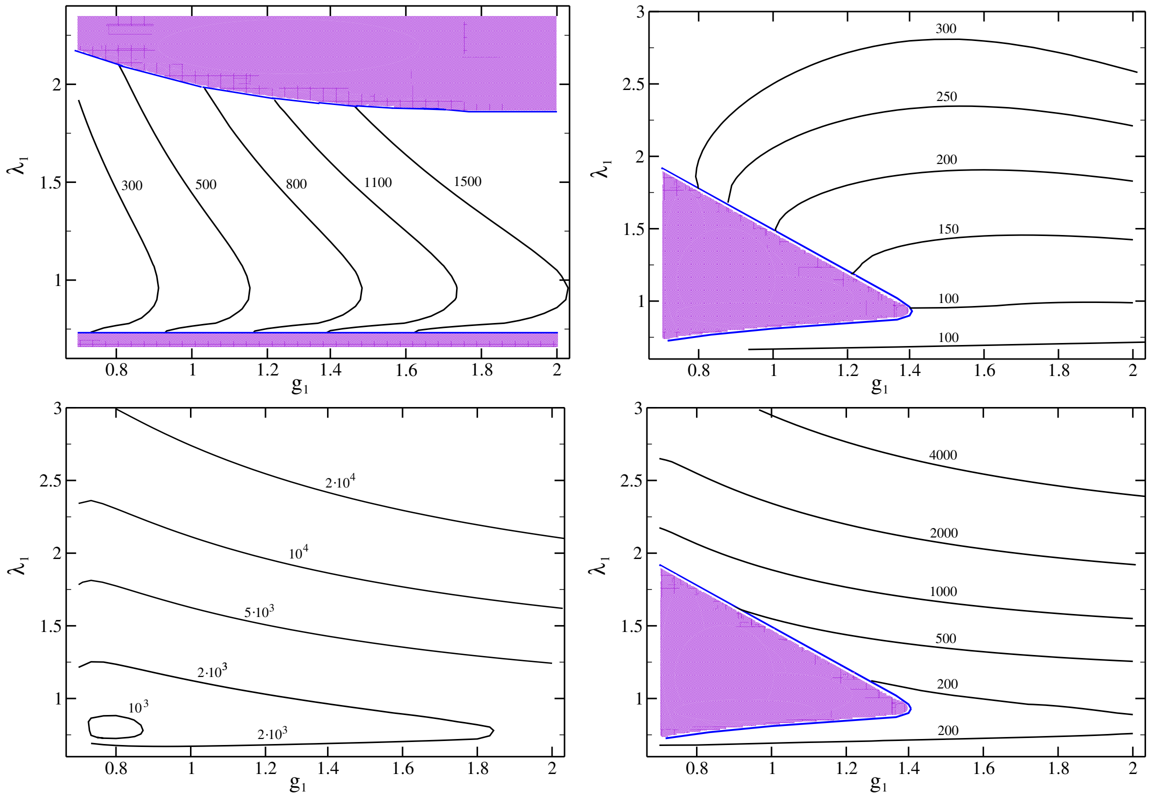

Figure 6: Final fine-tuning contours for the Little Higgs model of [14], using c_0 and c_0' as unknown parameters, for the two regions of solutions: a) (top) and b) (bottom), and two different values of the Higgs mass: $m_h = 115\,\text{GeV}$ (left) and $m_h = 250\,\text{GeV}$ (right).

4. A Little Higgs model with T-parity [15]

This model [15] is still based on the same SU(5)/ SO(5) structure of the Littlest Higgs model (with a gauged $[\mathrm{SU}(2) \times \mathrm{U}(1)]^2$ subgroup) and the gauge and scalar field content is the same, as described in appendix B.1 (although extended versions are possible [15]). However, the lagrangian is different: a T-parity is imposed such that the triplet and the heavy gauge bosons are T-odd while the Higgs doublet is T-even. This T-parity plays a role similar to R-parity in SUSY: it has the welcome effect of forbidding a number of dangerous couplings (like the $h^2\phi$ one responsible for the triplet VEV, as discussed in previous sections; or direct couplings of the SM fields to the new gauge bosons) improving dramatically the fit to electroweak data.

The gauge kinetic part of the lagrangian is as in eq. (B.8) but T-parity imposes the equalities

$$g_1 = g_2 = \sqrt{2} g\,, \qquad g_1' = g_2' = \sqrt{2} g'\,, \tag{4.1}$$

where g and g' are the gauge coupling constants of the SM. Imposing T-invariance on the fermionic sector requires the introduction of several new degrees of freedom, and the scalar operators of (B.11) are replaced by a T-symmetric expression given by (B.21).

The squared masses to $\mathcal{O}(h^2)$ in this model are similar to those in the Littlest Higgs model. In the gauge boson sector they are exactly the same as in (2.3), with gauge couplings related by eq. (4.1). In the fermion sector, despite the inclusion of extra degrees of freedom, the only mass relevant for our purposes is that of the heavy Top which, to order h^2, remains the same as in the Littlest Higgs model [see eq. (2.3)]. The squared masses of the other fermions do not have an h-dependence [they can be relatively heavy (in the multi-TeV range) and are irrelevant for low-energy phenomenology].

In the scalar sector, an important difference with respect to the Littlest Higgs model is that now there is no ϕh^2-coupling. As a result, the Higgs quartic coupling does not get modified after decoupling the triplet field and is simply given by:

$$\lambda = \frac{1}{4}(\lambda_a + \lambda_b)\,, \tag{4.2}$$

[now $\lambda_a = 2c(g^2 + g'^2) - c'\lambda_1^2$ and $\lambda_b = 2c(g^2 + g'^2)$] to be compared with eq. (2.8) for the Littlest Higgs. Another direct consequence of not having a ϕh^2-coupling is the absence of the off-diagonal entries in the scalar mass matrices in the CP-even, CP-odd and charged sectors (see appendix B.3 for details).

The one-loop-generated Higgs mass parameter, m^2, is given by the same expression as that of the Littlest Higgs model [eq. (2.15)] but, as we have seen, T-parity imposes strong relations between the parameters of the model. In particular, we have now

$$M_{W'}^2 = g^2 f^2\,, \qquad M_{B'}^2 = \frac{1}{5}g'^2 f^2\,, \qquad M_\phi^2 = 4\lambda f^2\,. \tag{4.3}$$

The model is therefore much more constrained than the Littlest Higgs.

For the fine-tuning analysis, we start by identifying the input parameters, which are now

$$p_i = \{\lambda_1,\ \lambda_2,\ c,\ c',\ f\}\,, \tag{4.4}$$

to be compared with (2.19) and (3.1). Again, we can leave f aside as explained after (2.19). The couplings $\lambda_{1,2}$ are related by the usual top-Yukawa constraint in eq. (2.2) while c and c' are related to λ through eq. (4.2). For a given value of the Higgs mass (and therefore of the coupling λ) the minimization condition for electroweak breaking can be solved for M_T^2, which fixes $\lambda_1^2 + \lambda_2^2$, but not λ_1 or λ_2 separately. From this continuum of solutions, the top mass constraint [eq. (2.2)] leaves only two of them, simply related by $\lambda_1 \leftrightarrow \lambda_2$. We will refer to these two solutions as

$$1)\ \lambda_1 \leq \lambda_2\,, \qquad 2)\ \lambda_2 \leq \lambda_1\,. \tag{4.5}$$

If λ is small, M_ϕ is not large enough to compensate the negative heavy Top contribution to the one-loop Higgs mass and the minimization condition is not satisfied. If, on the other hand, λ is too large then the Top contribution, which cannot be arbitrarily large (it grows with M_T, but only up to $M_T = \Lambda$), is also unable to satisfy the minimization condition. Thus, we obtain a limited range for m_h: $280\,\text{GeV} \lesssim m_h \lesssim 625\,\text{GeV}$, for $f = 1\,\text{TeV}$. This result has interest of itself for the phenomenology of the Littlest Higgs model with T-parity, with the caveat that possible two-loop (or 'tree-level') contributions to the Higgs mass parameter can change the limits of that interval for m_h, as we discuss in more detail below.

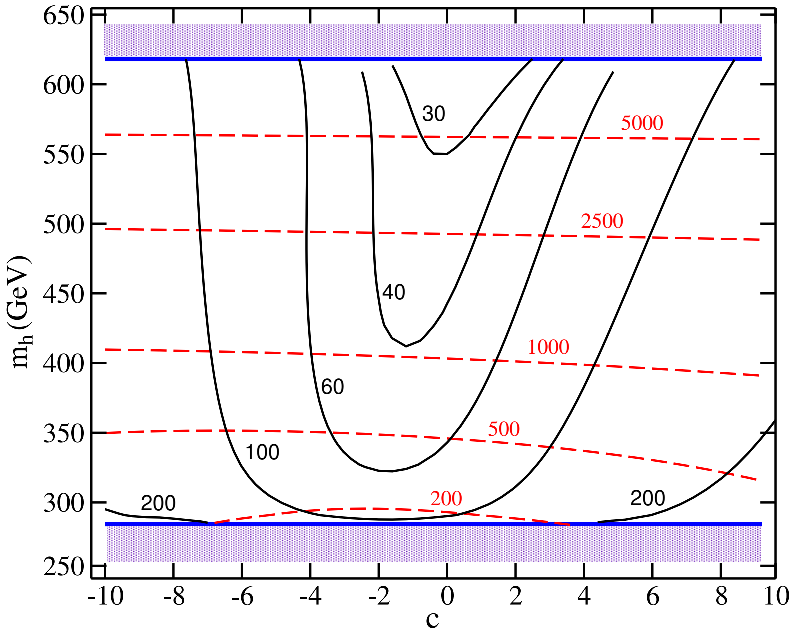

Figure 7: Fine-tuning contours in a Little Higgs model with T-parity, using c_0 and c'_0 of eq. (2.7) as unknown parameters. Solid (dashed) lines correspond to case 1 (2) of eq. (4.5).

The resulting constrained fine-tuning [using c_0 and c'_0 of eq. (2.7) as unknown parameters] is shown in figure 7. As g_1 is not a free-parameter anymore, we present our results in the plane $\{c, m_h\}$. The black solid lines correspond to case 1) and the red dashed ones to case 2). At the lower bound for m_h, which is determined by the minimal possible value of $M_T^2 = (\lambda_1^2 + \lambda_2^2) f^2$, one has $\lambda_1 = \lambda_2 = \lambda_t$ and therefore cases 1) and 2) give the same results for the fine-tuning, as can be seen in the figure. At the upper bound on m_h one has $M_T^2 = \Lambda^2$, which implies $\lambda_i \simeq 4\pi$ for $i = 1$ or 2, at the limit of perturbativity. We see that the fine-tuning is sizeable throughout all parameter space in spite of the large values of the Higgs mass. It is always larger for case 2) because a larger value of λ_1 affects directly the parameter λ_a and therefore the value of λ. In fact, as will be clearer shortly, the largest contribution to the fine-tuning comes, in most cases, through the dependence of λ on c, c' and λ_1.

From the previous discussion, it follows that at some future time, after the Higgs mass has already been measured (and thus λ gets fixed), the fine-tuning would get dramatically reduced, especially in case 2). This is shown by figure 8, left plot, which presents the fine-tuning when the constraint of fixed λ is enforced. The fine-tuning is nearly independent of c, and varies only through the values of $\lambda_{1,2}$, getting the smallest values at the boundaries of parameter space. This can be understood from the simple analytical approximation

$$\Delta \simeq \frac{M_T^2}{2\lambda v^2} \frac{|\lambda_1^2 - \lambda_2^2|}{\sqrt{\lambda_1^4 + \lambda_2^4}} \frac{3\lambda_t^2}{2\pi^2} \log \frac{\Lambda^2}{M_T^2} \,, \tag{4.6}$$

which is easy to derive and explains why cases 1) and 2) give very similar values for the fine-tuning.[11] Although the fine-tuning is moderate, we still have to worry about the tuning

[11]The small sensitivity to c and the small difference between scenarios 1) and 2) which can be appreciated

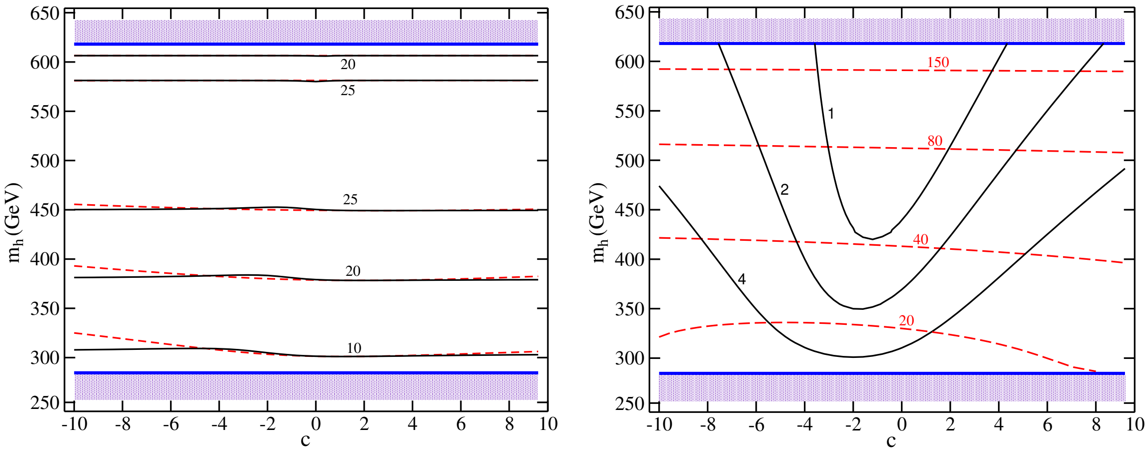

Figure 8: Left: Same as in figure 7 but keeping fixed λ. Right: Fine-tuning associated to λ itself. [Solid (dashed) lines correspond to case 1 (2) of eq. (4.5)].

in λ itself, as we did in section 3 for the model of ref. [14]. We show that tuning in the right plot of figure 8. Analytically we find

$$\Delta^{(\lambda)} \simeq \frac{\lambda_1^2}{4\lambda}\left[\frac{4c'^2\lambda_1^4}{\lambda_1^4+\lambda_2^4}+(c'-c_0')^2+16(c-c_0)^2\frac{(g^2+g'^2)^2}{\lambda_1^4}\right]^{1/2}. \tag{4.7}$$

We see that there is a big difference between cases 1) and 2). In case 1), the coupling λ_1 varies between λ_t at the lower limit of m_h and $\lambda_t/\sqrt{2}$ at the upper limit, and it does not cost much to get λ right. Therefore the associated tuning is always small. In case 2), λ_1 is of moderate size ($\sim \lambda_t$) near the lower limit on m_h but grows significantly when m_h increases (reaching $\lambda_1 \sim 4\pi$ near the upper limit). Then, getting λ right requires small values of c' and, being unnatural, this causes a sizeable tuning. Coming back to figure 7, one can easily check that the dependence of the fine-tuning in that plot on c and m_h can be understood as a particular combination of the two effects shown in figure 8.

Finally, let us consider the effect of two-loop (or 'tree-level') contributions to the Higgs mass parameter which, as mentioned, can allow Higgs masses below the (quite high) lower limit $m_h \geq 280\,\text{GeV}$ of figure 7. We mimic this effect by adding a constant mass term $1/2\mu_0^2h^2$ to the Higgs potential (allowing both signs of μ_0^2). From the arguments given in previous sections, we do not expect big changes in the fine-tuning but it is interesting to consider this possibility as a way of accessing regions of lower Higgs mass, which are more attractive phenomenologically. Notice that eq. (4.4) is now enlarged by one more parameter, namely μ_0^2. The resulting fine-tuning for cases 1) and 2) of eq. (4.5) is shown in figure 9, (left and right plots, respectively), setting $c = 0$ (which nearly minimizes the fine-tuning). For Higgs masses accessible already with $\mu_0 = 0$, the fine-tuning does not change much, as expected, while for lower Higgs masses the fine-tuning increases [case 1)] or remains large [case 2)]. We see that case 1) continues to be the best option.

in figure 8 is a subtle effect [not captured by the approximation (4.6)] due to the dependence of λ on c, c' and λ_1 (even though we are fixing λ). Such effects are discussed in appendix A.

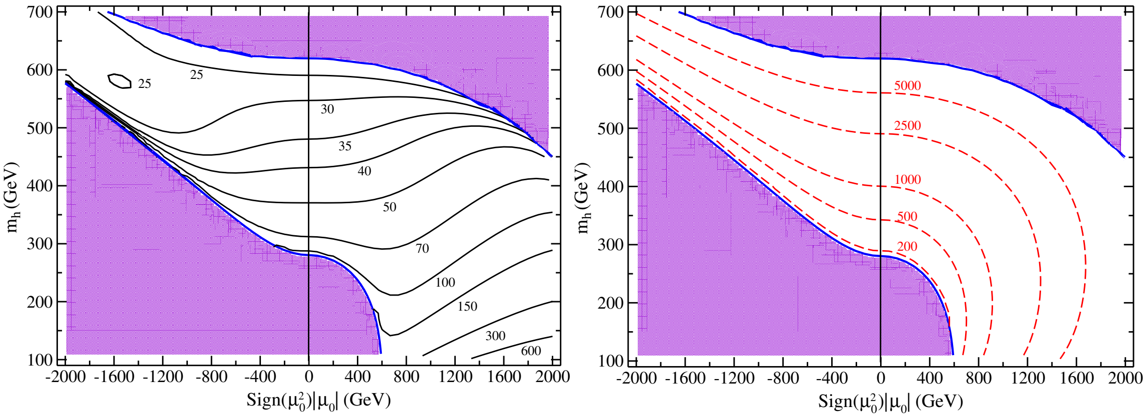

Figure 9: Fine-tuning contours in a Little Higgs model with T-parity, with a 'tree-level' μ_0^2 mass parameter, using c_0 and c_0' of eq. (2.7) as unknown parameters and setting $c = 0$. The left (right) plot corresponds to case 1 (2) of eq. (4.5).

Figures 7 and 9 summarize our results for the model analyzed in this section. As for the models of sections 2 and 3, the fine-tuning is always substantial ($\Delta > 10$) and usually comparable to (or higher than) that of the Little Hierarchy problem [$\Delta \gtrsim \mathcal{O}(100)$] and worse than in the MSSM. Notice also that the lowest fine-tuning, $\Delta \sim 25$, is obtained for large values of the Higgs mass, $m_h \gtrsim 500\,\mathrm{GeV}$, which is generically disfavoured from fits to precision electroweak observables [20]. In addition, such large values of m_h are less satisfactory from the point of view of the Little Higgs philosophy: the Little Higgs mechanism is interesting because it might explain the lightness of the Higgs compared to the TeV scale.

5. The Simplest Little Higgs model [16]

We now depart from the group structure of the Littlest Higgs and consider a model, proposed in [16], that is based on a global $[\mathrm{SU}(3)\times\mathrm{U}(1)]^2/[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$ (for an study of the phenomenology of the model see [23]). The initial gauged subgroup is $[\mathrm{SU}(3)\times\mathrm{U}(1)_X]$ which gets broken to the electroweak subgroup, with

$$\frac{1}{g'^2} = \frac{1}{3g^2} + \frac{1}{g_x^2}\,. \tag{5.1}$$

This symmetry breaking is triggered by the VEVs f_1 and f_2 of two SU(3) triplets, Φ_1 and Φ_2. For later use we define

$$f^2 \equiv f_1^2 + f_2^2\,, \tag{5.2}$$

which measures the total amount of breaking. This spontaneous breaking produces 10 Goldstone bosons, 5 of which are eaten by the Higgs mechanism to make massive a complex SU(2) doublet of extra W's, $(W'^{\pm}, W'^{0})$, and an extra Z'. The remaining 5 degrees of freedom are: H [an SU(2) doublet to be identified with the SM Higgs] and η (a singlet). Details about this breaking are left for appendix B.4. The initial tree-level lagrangian has a structure similar to eq. (2.1). In particular, m^2 and λ are zero at this level.

As in previous models, in order to study the electroweak breaking, we need to consider the one-loop Higgs potential, for which we have to to compute the h-dependent masses of the model. We collect here these masses leaving again details for appendix B.4. In the gauge sector, besides the massless photon, the rest of gauge bosons have the following masses. For the charged $(W^{\pm}, W'^{\pm})$ pair, one has, expanding in powers of h,

$$\begin{aligned} m^2_{W'^{\pm}}(h) &= M^2_{W'} - \frac{1}{4} g^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\ m^2_{W^{\pm}}(h) &= \frac{1}{4} g^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \end{aligned} \tag{5.3}$$

with

$$M^2_{W'} \equiv \frac{1}{2} g^2 f^2 . \tag{5.4}$$

For the (Z'^0, Z^0) pair,

$$\begin{aligned} m^2_{Z'^0}(h) &= M^2_{Z'} - \frac{1}{4}(g^2 + g'^2) h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\ m^2_{Z^0}(h) &= \frac{1}{4}(g^2 + g'^2) h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \end{aligned} \tag{5.5}$$

with

$$M^2_{Z'} \equiv \frac{2g^2}{3 - t_w^2} f^2 , \tag{5.6}$$

where $t_w \equiv g'/g$. Finally, the complex W'^0 has mass

$$m^2_{W'^0}(h) = M^2_{W'} . \tag{5.7}$$

The fermion sector is enlarged as usual. The states relevant for electroweak breaking are the SM top quark and a heavy Top, with masses squared

$$\begin{aligned} m^2_T(h) &= M^2_T - \frac{1}{2} \lambda_t^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\ m^2_t(h) &= \frac{1}{2} \lambda_t^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \end{aligned} \tag{5.8}$$

where

$$M^2_T \equiv \lambda_1^2 f_1^2 + \lambda_2^2 f_2^2 , \tag{5.9}$$

where $f_{1,2}$ are the triplet VEVs. Here $\lambda_{1,2}$ are new Yukawa couplings of the Little Higgs model, and λ_t is the SM top Yukawa coupling, given by the relation

$$\frac{f^2}{\lambda_t^2} = \frac{f_1^2}{\lambda_2^2} + \frac{f_2^2}{\lambda_1^2} = f_1^2 f_2^2 \left(\frac{1}{\lambda_1^2 f_1^2} + \frac{1}{\lambda_2^2 f_2^2} \right) . \tag{5.10}$$

One can trivially check the cancellation of h^2 terms in $\mathrm{Str} M^2$ from the explicit expressions of the masses given above. In fact, the cancellation holds to all orders in h (and η), as is clear from the more general formula for the masses presented in appendix B.4

[see eq. (B.34)]. Therefore, and in contrast with previous models, one-loop quadratically divergent corrections from gauge or fermion loops do not induce scalar operators to be added to the lagrangian. Then, no Higgs quartic coupling is present at this level.

Less divergent one-loop corrections do induce both a mass term and a quartic coupling for the Higgs. Using again the $\overline{\rm MS}$ scheme in Landau gauge[12] and setting the renormalization scale $Q = \Lambda$, it is straightforward to compute the one-loop potential including fermion and gauge boson loops once the masses are known as a function of h. Performing an expansion of this potential in powers of h, one gets [16]

$$V(h) = \frac{1}{2}\delta m^2 h^2 + \frac{1}{4}\left[\delta_1\lambda(h) - \frac{\delta m^2}{3}\frac{f^2}{f_1^2 f_2^2}\right] h^4 + \cdots \tag{5.11}$$

with

$$\begin{aligned}\delta m^2 = & \frac{3}{32\pi^2}\left[g^2 M_{W'}^2\left(\log\frac{\Lambda^2}{M_{W'}^2} + \frac{1}{3}\right) + \frac{1}{2}(g^2+g'^2)M_{Z'}^2\left(\log\frac{\Lambda^2}{M_{Z'}^2}+\frac{1}{3}\right)\right] - \\ & -\frac{3}{8\pi^2}\lambda_t^2 M_T^2\left(\log\frac{\Lambda^2}{M_T^2}+1\right) + \cdots ,\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\delta_1\lambda(h) = & -\frac{3}{128\pi^2}\left[g^4\left(\log\frac{M_{W'}^2}{m_W^2(h)} - \frac{1}{2}\right) + \frac{1}{2}(g^2+g'^2)^2\left(\log\frac{M_{Z'}^2}{m_Z^2(h)} - \frac{1}{2}\right)\right] + \\ & +\frac{3}{16\pi^2}\lambda_t^4\left(\log\frac{M_T^2}{m_t^2(h)} - \frac{1}{2}\right) + \cdots ,\end{aligned} \tag{5.13}$$

where the dots in (5.12) and (5.13) stand for subdominant contributions (in particular those from the η and the Higgs field itself, which was also subdominant in previous models).

The radiatively induced Higgs mass, δm^2, is dominated as usual by the negative heavy Top contribution, which is again too large (being $M_T^2 \geq 4\lambda_t^2 f_1^2 f_2^2/f^2$) and now there is no bosonic contribution that can be used to compensate it. This problem is solved [16] by adding to the tree-level potential a mass μ^2 for the triplets $\Phi_{1,2}$ (see appendix B.4). Such operator contributes to the Higgs potential the piece

$$\delta_0 V = \frac{1}{2}\mu_0^2 h^2 - \frac{1}{48}\frac{\mu_0^2 f^2}{f_1^2 f_2^2}h^4 + \cdots \tag{5.14}$$

where μ_0^2 is given in terms of the fundamental mass parameter μ^2 by

$$\mu_0^2 = \mu^2 \frac{f^2}{f_1 f_2} . \tag{5.15}$$

By choosing $\mu_0^2 > 0$ we get a positive contribution to the Higgs mass parameter that can compensate the heavy Top contribution in δm^2. The tree-level value of the Higgs quartic coupling from (5.14) is then negative but the large (and positive) radiative corrections in (5.13) can easily overcome that effect.

[12] Our scheme differs from that used in [16], but the difference is numerically small.

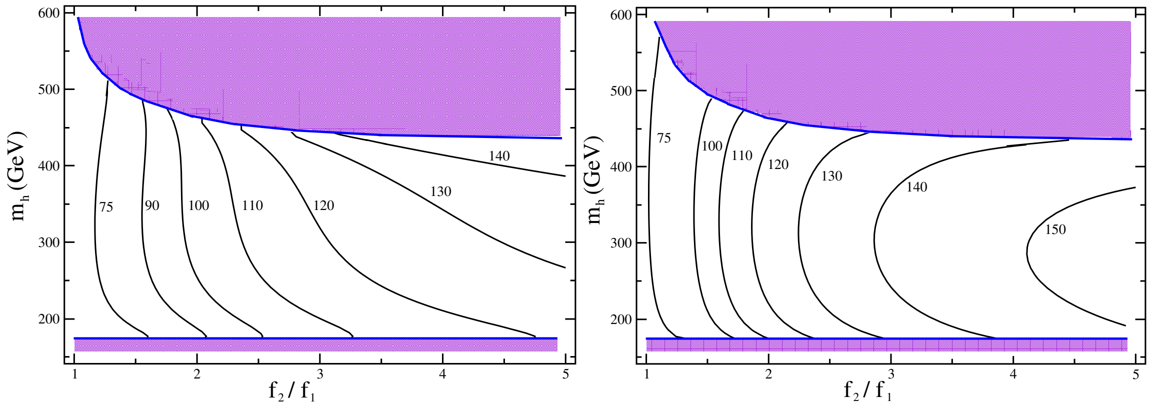

Figure 10: Fine-tuning contours for the Simplest Little Higgs model for cases 1) (left plot) and 2) (right plot) of eq. (5.18).

In order to compute the fine-tuning in this model we use the previous potential, (5.11) plus (5.14):

$$V(h) = \frac{1}{2}(\mu_0^2 + \delta m^2)h^2 + \frac{1}{4}\left[\delta_1\lambda(h) - \frac{f^2}{3f_1^2 f_2^2}\left(\delta m^2 + \frac{\mu_0^2}{4}\right)\right] h^4 + \cdots \tag{5.16}$$

As mentioned, it does not contain the subdominant contributions from η and the Higgs field. The input parameters are now:

$$\{\lambda_1, \lambda_2, \mu^2, f_1, f_2\}\,. \tag{5.17}$$

Without loss of generality we can choose $f_1 \leq f_2$, in which case the UV cut-off is $\Lambda = 4\pi f_1$. Since we want $\Lambda = 10\,\mathrm{TeV}$ (the scale of the Little Hierarchy problem) we also set $f_1 = 1\,\mathrm{TeV}$. As f_1 and f_2 are not the only mass scales in the problem (there is μ^2 as well) it is important to include the fine-tuning associated to them, which might be large now.

The Higgs mass that results from the potential (5.16), after trading μ_0^2 by v using the minimization condition, can be computed as a function of M_T^2 for fixed f_2/f_1. For any pair $\{\lambda_1, \lambda_2\}$ that gives a particular value of M_T^2, there is another pair $\{\lambda_1, \lambda_2\} \to \{\lambda_2 f_2/f_1, \lambda_1 f_1/f_2\}$ that gives the same M_T^2. Therefore each choice of M_T^2 (to get a particular value of m_h) corresponds to two different solutions in terms of $\lambda_{1,2}$. We will refer to them as

$$1)\ \lambda_1 f_1 \leq \lambda_2 f_2\,, \qquad 2)\ \lambda_1 f_1 \geq \lambda_2 f_2\,. \tag{5.18}$$

As mentioned above, these two solutions are related by the interchange $\lambda_1 f_1 \leftrightarrow \lambda_2 f_2$. Figure 10 gives the fine-tuning in the plane $\{m_h, f_2/f_1\}$ for these two cases

We see from these plots that the fine-tuning is sizeable and increases with f_2/f_1. From the bound $M_T \geq 2\lambda_t f_1 f_2/f$ and the fact that δm^2 and $\delta_1\lambda$ cannot be arbitrarily large, it follows that m_h^2 is limited to a certain range. This range depends on the value of f_2/f_1: for $f_2 = f_1$ one gets $163\,\mathrm{GeV} \leq m_h \leq 606\,\mathrm{GeV}$ and a narrower range for larger f_2/f_1, as can be seen in figure 10.

To access lower values of m_h one can add a piece λ_0 to the Higgs quartic coupling in the potential (5.16). This new term can result from the unknown heavy physics at the cut-off Λ. For $\lambda_0 < 0$ one can get values of m_h below the lower bounds discussed before. In the presence of such term we should also worry about the quadratically divergent contributions of scalars to the Higgs mass parameter. From

$$\delta V_1^{\rm quad} = \frac{\Lambda^2}{32\pi^2}(m_h^2 + 3m_G^2 + m_\eta^2)\,, \tag{5.19}$$

where m_h, m_G and m_η are the tree-level masses of the Higgs, the electroweak Goldstones and η respectively, one gets[13] (after substituting $\Lambda = 4\pi f_1$)

$$\delta_q m^2 = -\frac{5f^2}{8f_2^2}\mu_0^2 + 6\lambda_0 f_1^2\,. \tag{5.20}$$

The piece proportional to μ_0^2 is not particularly dangerous and can even be interpreted as a redefinition of the original μ_0^2 parameter, while the second term, proportional to the new coupling λ_0, can be sizeable, thus having a significant impact on the fine-tuning. In the presence of these quadratically divergent corrections we expect to have a contribution to the Higgs mass parameter of order $6\lambda_0 f_1^2$ already at the cut-off. Therefore we introduce such mass term in the potential, multiplied by some unknown coefficient c, from the beginning. As we did in previous models, we then split c into an unknown 'tree-level' contribution c_0 and a calculable radiative one-loop correction c_1, with $c = c_0 + c_1 = c_0 + 1$. Our potential is now

$$V(h) = \frac{1}{2}[\mu_0^2 + \delta m^2 + 6(c_0+1)\lambda_0 f_1^2]h^2 + \frac{1}{4}\left[\lambda_0 + \delta_1\lambda(h) - \frac{f^2}{3f_1^2 f_2^2}\left(\delta m^2 + \frac{\mu_0^2}{4}\right)\right]h^4 + \cdots \tag{5.21}$$

and the set of input parameters is enlarged to

$$\{\lambda_0, \lambda_1, \lambda_2, \mu^2, f_1, f_2, c_0\}\,. \tag{5.22}$$

Figure 11 shows the fine-tuning associated to this modified potential in the plane $\{c_0, \lambda_1\}$ for $m_h = 115$ Gev (left plot) and $m_h = 250\,\text{GeV}$ (right plot) for $f_2 = f_1$. As expected, lower Higgs masses can now be reached, but there is a fine-tuning price to pay. As shown by the right plot, in the case of larger Higgs masses, already accessible for $\lambda_0 = 0$, the effect of the new parameters c_0 and λ_0 allows the fine-tuning to be reduced if such parameters are chosen appropriately, but the effect is never dramatic (for the sake of comparison, we show by a dashed line, the fine-tuning corresponding to $\lambda_0 = 0$). However, the fine-tuning gets worse in most of the parameter space.

From figures 10 and 11, we can conclude that the fine-tuning in the Simplest LH model is similar to that of the models analyzed in previous sections: it is always significant and usually comparable to (or higher than) that of the Little Hierarchy problem [$\Delta \gtrsim \mathcal{O}(100)$].

[13] Of course, this contribution is due to the fact that the Simplest model does not include additional fields to cancel the quadratic divergencies from loops of its scalar fields.

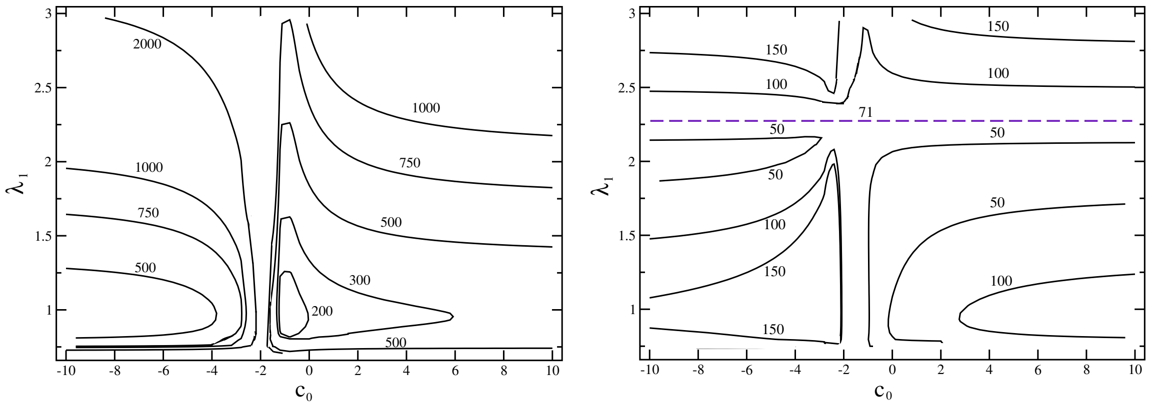

Figure 11: Fine-tuning contours for the Simplest Little Higgs model augmented by a 'tree-level' quartic coupling λ_0, with $m_h = 115\,\text{GeV}$ (left plot) and $m_h = 250\,\text{GeV}$ (right plot).

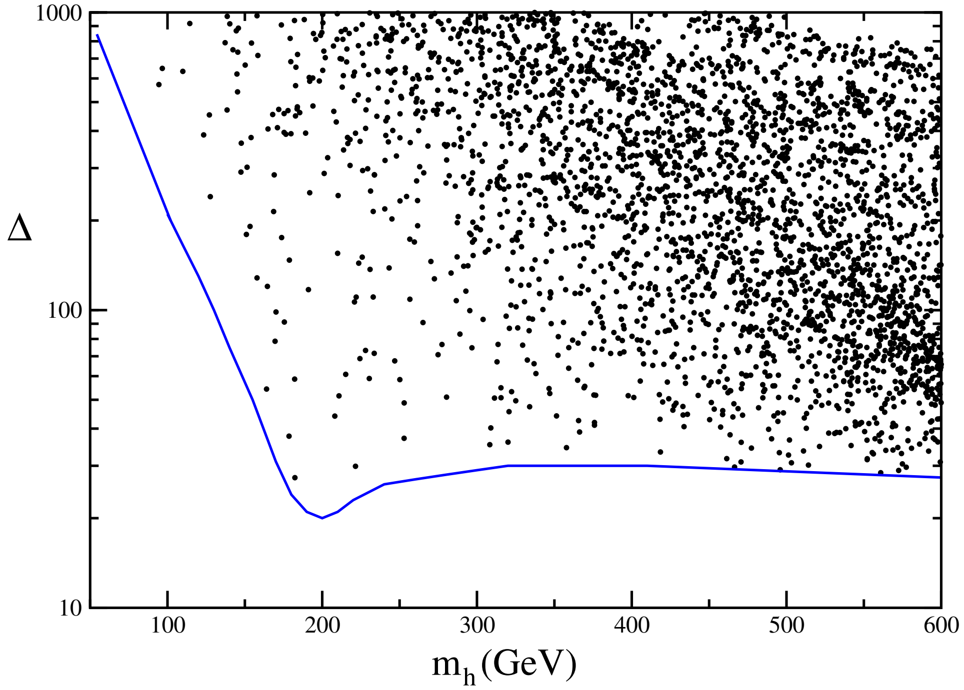

Figure 12: Scatter-plot of the fine-tuning in the Simplest Little Higgs model as a function of the Higgs mass.

Only for some small regions of parameter space is Δ comparable to the MSSM one ($\Delta \sim 20-40$ for $m_h \lesssim 125\,\text{GeV}$); usually it is much worse. The last point is illustrated by the scatter-plot of figure 12, which shows the value of Δ vs. m_h for random values of the parameters (5.22) compatible with $v = 246\,\text{GeV}$. More precisely, we have set $f_1 = f_2 = 1\,\text{TeV}$ and chosen at random $\lambda_0 \in [-2, 2]$, $\lambda_1 \in [\lambda_t/\sqrt{2}, 15]$ and $c_0 \in [-10, 10]$. The solid line gives the minimal value of Δ as a function of m_h and has been computed independently (rather than deduced from the scatter plot). Clearly, the density of points gets sparser near this lower bound.

6. Conclusions

We have rigorously analyzed the fine-tuning associated to the electroweak breaking process in Little Higgs (LH) scenarios, focusing on four popular and representative models, corresponding to refs. [13]–[16].

Although LH models solve parametrically the Little Hierarchy problem [generating a Higgs mass parameter of order $f/(4\pi)$], our first conclusion is that these models generically have a substantial fine-tuning built-in, usually much higher than suggested by the rough considerations commonly made. This is due to implicit tunings between parameters that can be overlooked at first glance but show up in a more systematic analysis. This does not demonstrate, of course, that all LH models are necessarily fine-tuned, but it stresses the need of a rigorous analysis in order to claim that a particular model is not fine-tuned, especially if a quantitative statement is attempted (e.g. to compare its degree of fine-tuning with that of the MSSM). In this respect, the analysis presented here can also be helpful as a guide to the ingredients that typically increase the fine-tuning in LH models, in order to correct them in improved constructions.

We have quantified the degree of fine-tuning following the 'standard' criterion of Barbieri and Giudice [6], through a fine-tuning parameter Δ, that can be computed in each model ($\Delta \simeq 100$ means a fine-tuning at the one percent level, etc.), finding that the four LH scenarios analyzed here present fine-tuning ($\Delta > 10$) in all cases. The results are summarized in the plots of figures 3 (for the Littlest Higgs), 6 (for the modified Littlest Higgs), 7 and 9 (for the Littlest Higgs with T-parity), and 10 and 11 (for the Simplest Little Higgs). Actually, the fine-tuning is comparable to or higher than –sometimes much higher– than the one associated to the Little Hierarchy problem of the SM (given by the blue line of figure 1) in most of the parameter space of these models. Since LH models have been designed to solve the Little Hierarchy problem, we believe this is a serious drawback. Likewise, the fine-tuning is usually worse than that of supersymmetric models ($\Delta = 20-40$ for the MSSM and lower for other supersymmetric scenarios), which succeed at stabilizing a much larger hierarchy ($\Lambda \simeq M_{GUT}$ or M_{Planck} rather than $\Lambda \simeq 10\,\mathrm{TeV}$).

We can make the previous statements more precise. Figure 13 shows the fine-tuning Δ as a function of m_h for different scenarios. The curve labelled "SM" represents the fine-tuning of the Little Hierarchy problem in the SM, with a common cut-off $\Lambda = 10\,\mathrm{TeV}$, as discussed in the introduction. The "MSSM" line shows the fine-tuning of the MSSM.[14] Then, for each LH model analyzed in sects. 2–4 we have plotted (lines labeled "Littlest", "Littlest 2", "T-parity" and "Simplest") the minimum value of Δ accessible by varying the parameters of the model. Usually, only in a quite small area of parameter space of each model is the fine-tuning close to the lower bound shown, so the LH curves in figure 13 are a very conservative estimate of the fine-tuning in the corresponding LH models. This point is illustrated by figure 12 for the Simplest LH model (the best behaved): the lower line in that plot corresponds to the "Simplest" line in figure 13. Now we see that the value of Δ

[14] This curve has been obtained for large $\tan\beta$ (which minimizes the fine-tuning) but disregarding stop-mixing effects (which can help in reducing the fine-tuning). It also takes into account the most recent experimental value for the top mass [25], which makes the fine-tuning lower than in previous analyses.

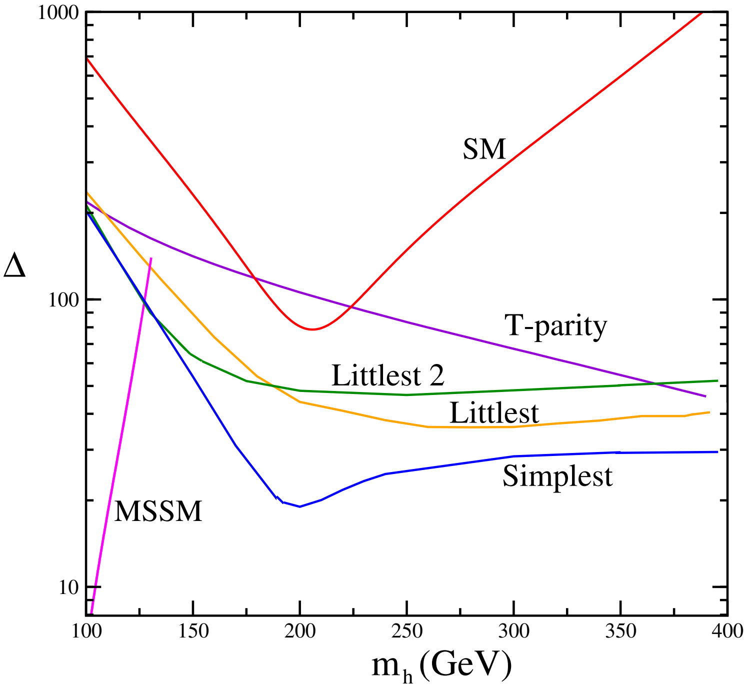

Figure 13: Comparative summary of the fine-tuning vs. m_h for different scenarios. The SM curve corresponds to a common cut-off $\Lambda = 10\,\mathrm{TeV}$. The curves for Little Higgs models (lines labeled "Littlest", "Littlest 2", "T-parity" and "Simplest") are lower bounds on the corresponding fine-tuning, see text for details.

for all these models is $\geq \mathcal{O}(100)$ in most of parameter space, and larger that $20-30$ in all cases. This fine-tuning is larger than the MSSM one, at least for the especially interesting range $m_h \lesssim 130\,\mathrm{GeV}$. Notice here that $m_h \gtrsim 135\,\mathrm{GeV}$ is not available in the MSSM if the supersymmetric masses are not larger than $\sim 1\,\mathrm{TeV}$. This limitation does not hold for other supersymmetric models, e.g. those with low-scale SUSY breaking, as discussed in ref. [11], which are definitely in better shape than LH models concerning fine-tuning issues.

In our analysis we have chosen $f = 1\,\mathrm{TeV}$ ($f_1 = 1\,\mathrm{TeV}$ in the Simplest model) which roughly corresponds to the scale of new particles introduced in Little Higgs models to stabilize the little hierarchy between the electroweak scale and a cut-off scale at $\Lambda \sim 4\pi f \sim 10\,\mathrm{TeV}$. Some of the models studied are in trouble with precision electroweak fits for $f = 1\,\mathrm{TeV}$ and require larger values of f [26]. As we discussed, the fine-tuning increases roughly like f^2 and combining fine-tuning analyses with precision electroweak fits will make those models even less appealing.

Regarding the specific ingredients that potentially increase the fine-tuning in LH models, we stress two of them. First, the LH lagrangian is generically enlarged with operators that have the same structure as those generated through the quadratically divergent radiative corrections to the potential (and are necessary for the viability of the models). Such operators have two contributions: the radiative one (calculable) and the 'tree-level' one (arising from physics beyond the cut-off and unknown). Very often the required value of the coefficient in front of a given operator is much smaller than the calculable contribution, which implies a tuning (usually unnoticed) between the tree-level and the one-loop pieces (similar to the hierarchy problem in the SM). Second, the value of the Higgs quartic coupling, λ, receives several contributions which have a non-trivial dependence on the various parameters of the model. Sometimes it is difficult, without an extra fine-tuning, to keep λ small, as required to have m_h in the region that is more interesting phenomenologically.

Acknowledgments

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A. Fine-tuning estimates with constraints

Let $F(x_i)$ be a quantity that depends on some input parameters x_i ($i = 1, \ldots, N$), considered as independent. The fine-tuning in F associated to x_i is Δ_i, defined by

$$\frac{\delta F}{F} = \Delta_i \frac{\delta x_i}{x_i} \,. \tag{A.1}$$

It is convenient for the following discussion to switch to vectorial notation and define

$$\vec{\Delta} F \equiv \left\{ \frac{\partial \log F}{\partial \log x_i} \right\} , \tag{A.2}$$

which is a vector of dimension N with components Δ_i, and is simply the gradient of $\log F$ in the $\{\log x_i\}$ space. We define the total fine-tuning associated to the quantity F as

$$\Delta F \equiv \left[\sum_i \Delta_i^2 \right]^{1/2} = ||\vec{\Delta} F|| \,. \tag{A.3}$$

This definition can be interpreted in the following statistical sense: if the input parameters x_i are allowed to vary (randomly and independently) around their values, with $\delta x_i / x_i$ following gaussian distributions of width σ_i, then $\delta F/F$ in eq. (A.1) follows a gaussian distribution of width (squared)

$$\sigma_F^2 = \sum_i \Delta_i^2 \sigma_i^2 \,, \tag{A.4}$$

which can be taken as a measure of the fine-tuning in F: notice that the quantity $(\delta F)^2$ is expected to take a value of order $F^2 \sigma_F^2$. Now, it is sensible to assume that an unknown parameter x_i may vary within a range of order x_i (this is a common assumption in fine-tuning discussions) and hence to take $\sigma_i \sim 1$. Then, the definition in eq. (A.3) follows. In the case of an input parameter which is measured with some non-negligible experimental uncertainty $\epsilon_i^{\exp}$ one should use $\sigma_i = \epsilon_i^{\exp}/x_i$ in eq. (A.4). This is equivalent to rescaling the corresponding Δ_i as $\Delta_i \to \Delta_i \sigma_i$ [8] in eq. (A.3). (This prescription was applied to the top Yukawa coupling in [1].)

Next suppose that the x_i are not independent but are instead related by a number of (experimental or theoretical) constraints $G_\alpha^{(0)}(x_i) = 0$ ($\alpha = 1, \ldots, m$ with $m < N$) so that, when one computes the fine-tuning in F, one is only free to vary the input x_i's in such a way that the constraints are respected. In order to compute the "constrained fine-tuning" in F we first define, for each constraint, the vector $\vec{\Delta} G_\alpha^{(0)} = \{\partial G_\alpha^{(0)}/\partial \log x_i\}$ which is normal to the $G_\alpha^{(0)} = 0$ hypersurface in the $\{\log x_i\}$ space. We then use the Gramm-Schmidt procedure to get from the vectors $\vec{\Delta} G_\alpha^{(0)}$ an orthonormal set, $\vec{\Delta} G_\alpha$, that satisfies

$$\vec{\Delta} G_\alpha \cdot \vec{\Delta} G_\beta = \delta_{\alpha\beta}\,. \tag{A.5}$$

Then we can find the constrained fine-tuning simply projecting the unconstrained $\vec{\Delta} F$ on the $G_\alpha = 0$ manifold [which coincides with the $G_\alpha^{(0)} = 0$ manifold]:

$$\vec{\Delta} F\Big|_G = \vec{\Delta} F - \sum_\alpha (\vec{\Delta} F \cdot \vec{\Delta} G_\alpha) \vec{\Delta} G_\alpha\,. \tag{A.6}$$

Finally,

$$\Delta F|_G = \left|\left| \vec{\Delta} F\Big|_G \right|\right| = \left[(\Delta F)^2 - \sum_\alpha (\vec{\Delta} F \cdot \vec{\Delta} G_\alpha)^2 \right]^{1/2}\,. \tag{A.7}$$

As was to be expected, the constrained fine-tuning, $\Delta F|_G$, is always smaller that the unconstrained fine-tuning ΔF.

The previous procedure can also be seen as a change of coordinates in the "euclidean" $\{\log x_i\}$ space [which leaves eq. (A.3) invariant], such that the first m new coordinates $\{\log y_\alpha\}$ span the same subspace as the $\vec{\Delta} G_\alpha^{(0)}$ vectors. These m coordinates have to be simply eliminated from eq. (A.3), as they are fixed by the constraints, while the remaining ones are totally unconstrained. In this way the final expression (A.7) is recovered.

Note that if F does not depend on some of the parameters, say $\{x_a\}$, but some of the constraints do, the constrained fine-tuning will generically depend on the value of $\{x_a\}$, even if the other parameters remain the same. This is in fact a perfectly logical result. Notice that the fine-tuning quantity, ΔF, measures the relative change of F against the relative changes in the x_i parameters. Imagine a function $F = F(x_1)$ and a constraint $G^{(0)} = x_1 + x_2 + x_3 - C = 0$. If $x_2, x_3 \ll x_1$ the value of x_1 is essentially fixed and thus $\Delta F|_G$ should be small (if x_2, x_3 are allowed to change a 100%, x_1 is only allowed to change in a very small relative range). In the opposite case, if $x_2, x_3 \gg x_1$ (for the same value of x_1) the x_1 parameter can be freely varied and thus $\Delta F|_G \simeq \partial \log F/\partial \log x_1$. Therefore, $\Delta F|_G$ does depend on x_2 and x_3 even if $F = F(x_1)$. We have found this effect in some of the scenarios studied (although it always had a mild impact on the final fine-tuning); see section 4, footnote 11.

B. Formulas for Little Higgs models

B.1 The Littlest Higgs model

This model [13] is based on an SU(5)/ SO(5) nonlinear sigma model. The spontaneous breaking of SU(5) down to SO(5) is produced by the vacuum expectation value of a 5×5

symmetric matrix field Φ. We follow [13] and choose

$$\langle\Phi\rangle = \Sigma_0 = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & 1 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{B.1}$$

This breaking of the global SU(5) symmetry produces 14 Goldstone bosons which include the Higgs doublet field. These Goldstone bosons can be parametrized through the nonlinear sigma model field

$$\Sigma = e^{i\Pi/f}\Sigma_0 e^{i\Pi^T/f} = e^{2i\Pi/f}\Sigma_0\,, \tag{B.2}$$

with $\Pi = \sum_a \Pi^a X^a$, where Π_a are the Goldstone boson fields and X^a the broken SU(5) generators. The model assumes a gauged $\mathrm{SU}(2)_1 \times \mathrm{U}(1)_1 \times \mathrm{SU}(2)_2 \times \mathrm{U}(1)_2$ subgroup of SU(5) with generators (σ^a are the Pauli matrices)

$$Q_1^a = \begin{pmatrix} \sigma^a/2 & 0_{2\times 3} \\ 0_{3\times 2} & 0_{3\times 3} \end{pmatrix}, \qquad Q_2^a = \begin{pmatrix} 0_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & -\sigma^{a^*}/2 \end{pmatrix}, \tag{B.3}$$

and

$$Y_1 = \frac{1}{10}\mathrm{diag}(-3,-3,2,2,2)\,, \qquad Y_2 = \frac{1}{10}\mathrm{diag}(-2,-2,-2,3,3)\,. \tag{B.4}$$

The vacuum expectation value in eq. (B.1) breaks $\mathrm{SU}(2)_1 \times \mathrm{U}(1)_1 \times \mathrm{SU}(2)_2 \times \mathrm{U}(1)_2$ down to the diagonal $\mathrm{SU}(2) \times \mathrm{U}(1)$, identified with the SM group.

The Goldstone and (pseudo)-Goldstone bosons in the hermitian matrix Π in Σ fall in representations of the SM group as

$$\Pi = \begin{pmatrix} \xi & \frac{H^\dagger}{\sqrt{2}} & \phi^\dagger \\ \frac{H}{\sqrt{2}} & 0 & \frac{H^*}{\sqrt{2}} \\ \phi & \frac{H^T}{\sqrt{2}} & \xi^T \end{pmatrix} + \frac{1}{\sqrt{20}}\zeta^0\mathrm{diag}(1,1,-4,1,1)\,, \tag{B.5}$$

where $H = (h^0, h^+)$ is the Higgs doublet; ϕ is a complex SU(2) triplet given by the symmetric 2×2 matrix:

$$\phi = \begin{bmatrix} \phi^0 & \frac{1}{\sqrt{2}}\phi^+ \\ \frac{1}{\sqrt{2}}\phi^+ & \phi^{++} \end{bmatrix}, \tag{B.6}$$

the field ζ^0 is a singlet which is the Goldstone associated to the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \to \mathrm{U}(1)_Y$ breaking and finally, ξ is the real triplet of Goldstone bosons associated to $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \to \mathrm{SU}(2)$ breaking:

$$\xi = \frac{1}{2}\sigma^a\xi^a = \begin{bmatrix} \frac{1}{2}\xi^0 & \frac{1}{\sqrt{2}}\xi^+ \\ \frac{1}{\sqrt{2}}\xi^- & -\frac{1}{2}\xi^0 \end{bmatrix}. \tag{B.7}$$

All the fields in Π as written above are canonically normalized.

The kinetic part of the lagrangian is

$$\mathcal{L}_{kin} = \frac{f^2}{8}\mathrm{Tr}[(D_\mu\Sigma)(D^\mu\Sigma)^\dagger]\,, \tag{B.8}$$

where

$$D_\mu\Sigma = \partial_\mu\Sigma - i\sum_{j=1}^{2} g_j W_j^a (Q_j^a\Sigma + \Sigma Q_j^{aT}) - i\sum_{j=1}^{2} g'_j B_y (Y_j\Sigma + \Sigma Y_j^T)\,. \tag{B.9}$$

In this model, additional fermions are introduced in a vector-like coloured pair t', t'^c to cancel the Higgs mass quadratic divergence from top loops (other Yukawa couplings are neglected). The relevant part of the lagrangian containing the top Yukawa coupling is given by

$$\mathcal{L}_f = \frac{1}{2}\lambda_1 f \epsilon_{ijk}\epsilon_{xy}\chi_i\Sigma_{jx}\Sigma_{ky}u'^c + \lambda_2 f t' t'^c + h.c., \tag{B.10}$$

where $\chi_i = (t, b, t')$, indices i, j, k run from 1 to 3 and x, y from 4 to 5, and ϵ_{ijk} and ϵ_{xy} are the completely antisymmetric tensors of dimension 3 and 2, respectively.

As mentioned in the text, by considering gauge and fermion loops one sees that the lagrangian should also include gauge invariant terms of the form,

$$\begin{aligned} -\Delta\mathcal{L} = V \equiv\; & c\,\mathcal{O}_V(\Sigma) + c'\,\mathcal{O}_F(\Sigma) \\ =\; & cf^4\sum_{i=1,2} g_i^2 \sum_a \mathrm{Tr}[(Q_i^a\Sigma)(Q_i^a\Sigma)^*] + cf^4\sum_{i=1,2} g_i'^2\mathrm{Tr}[(Y_i\Sigma)(Y_i\Sigma)^*] - \\ & -\frac{1}{8}c' f^4\lambda_1^2\epsilon^{wx}\epsilon_{yz}\Sigma_{iw}\Sigma_{jx}\Sigma^{iy*}\Sigma^{jz*}\,, \end{aligned} \tag{B.11}$$

with c and c' assumed to be constants of $\mathcal{O}(1)$. The analysis of the spectrum and Higgs potential for this model is presented in section 2, after eq. (2.6).

B.2 A modified version of the Littlest Higgs model

This model is also based on the SU(5)/SO(5) Littlest Higgs [13], but modified [14] in such a way that only one abelian U(1) factor (identified with hypercharge) is gauged. The $SU(2)_1 \times SU(2)_2$ generators are as in the Littlest model [eq. (B.3)] and the hypercharge generator is $Y = \mathrm{diag}(1, 1, 0, -1, -1)/2$. The field content of the hermitian matrix Π in Σ is the same as in the Littlest Higgs model but now the field ζ^0 [Goldstone associated to the breaking of the U(1) symmetry left ungauged] is not absorbed by the Higgs mechanism (there is no B' now) and remains in the physical spectrum. In any case, this field plays no significant role in the discussion (it can be given a small mass to avoid phenomenological problems by adding explicit breaking terms [4]).

The kinetic part of the lagrangian is as in the Littlest Higgs, eq. (B.8) model but now with

$$D_\mu\Sigma = \partial_\mu\Sigma - i\sum_{j=1}^{2} g_j W_j^a (Q_j^a\Sigma + \Sigma Q_j^{aT}) - ig' B_Y (Y\Sigma + \Sigma Y^T)\,. \tag{B.12}$$

The fermionic couplings in the lagrangian can be kept as in the Littlest Higgs model also. Then the scalar operators $\mathcal{O}_F(\Sigma)$ and $\mathcal{O}_V(\Sigma)$, induced by fermion and gauge boson loops have the same form of eq. (B.11) but with the U(1) part limited to $U(1)_Y$ only. The main difference with respect to the Littlest Higgs case is that now the Higgs boson gets a small tree level mass of order $g'^2 f^2$ through the $\mathcal{O}_V(\Sigma)$ operator.

The h-dependent field masses, needed for the calculation of the one-loop Higgs potential, are the following. In the gauge boson sector we have

$$m^2_{W'}(h) = \frac{1}{4}(g_1^2+g_2^2)f^2 - \frac{1}{4}g^2h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \tag{B.13}$$

with no B' gauge boson. In the fermion sector, the heavy Top has mass

$$m^2_T(h) = M_T^2 + \mathcal{O}(h^2) = (\lambda_1^2+\lambda_2^2)f^2 - \frac{1}{2}\lambda_t^2h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \tag{B.14}$$

In the scalar sector, decomposing $h^0 \equiv (h^{0r}+ih^{0i})/\sqrt{2}$ and $\phi^0 \equiv i(\phi^{0r}+i\phi^{0i})/\sqrt{2}$ and using $\lambda'_a \equiv cg_2^2 - c'\lambda_1^2$ and $\lambda'_b \equiv cg_1^2$, combined in $\lambda'_+ \equiv \lambda'_a + \lambda'_b$ and $\lambda'_- \equiv \lambda'_a - \lambda'_b$, the masses are as follows. Writing simultaneously the relevant part of the mass matrices in the CP-even sector (using the basis $\{h^{0r},\phi^{0r}\}$), the CP-odd sector (in the basis $\{h^{0i},\phi^{0i}\}$) and the charged sector (in the basis $\{h^+,\phi^+\}$), we get

$$\begin{aligned} M^2_\kappa(h) = &\begin{bmatrix} \frac{1}{4}a_\kappa\lambda'_+h^2 + \frac{1}{\sqrt{2}}s_\kappa\lambda'_-ft + \mathcal{O}(h^4/f^2) & b_\kappa\lambda'_-fh + \mathcal{O}(h^2) \\ b^*_\kappa\lambda'_-fh + \mathcal{O}(h^2) & \lambda'_+\left(f^2 - c_\kappa h^2\right) + \mathcal{O}(h^4/f^2) \end{bmatrix} + \\ &+cg'^2\begin{bmatrix} f^2 - d_\kappa h^2 + \mathcal{O}(h^4/f^2) & \mathcal{O}(h^2) \\ \mathcal{O}(h^2) & 4f^2 - e_\kappa h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right) \end{bmatrix}, \end{aligned} \tag{B.15}$$

where the index $\kappa = \{0r, 0i, +\}$ labels the different sectors. The numbers a_κ, b_κ, c_κ and s_κ are as in (2.10) while $d_\kappa = \{1, 1/6, 1/6\}$ and $e_\kappa = 13|b_\kappa|^2/3$. We have also included in these mass matrices the contribution of the triplet VEV, $t \equiv \langle\phi^{0r}\rangle$, with

$$t \simeq -\frac{1}{2\sqrt{2}}\frac{\lambda'_-h^2}{(\lambda'_+ + 4cg'^2)f}. \tag{B.16}$$

As in the Littlest Higgs model, the off-diagonal entries in (B.15) are due to the $h^2\phi$ coupling which causes mixing between h and ϕ after electroweak symmetry breaking. This effect is negligible for the heavy triplet [at order h^2 in the masses, the components ϕ^{0r} and ϕ^{0i} can still be combined in a complex field ϕ^0]. We call h'^{0r}, h'^{0i} and h'^+ the light mass eigenvalues of (2.10) in the different sectors. The explicit masses for the different components of the triplet field are then[15]

$$\begin{bmatrix} m^2_{\phi^0}(h) \\ m^2_{\phi^+}(h) \\ m^2_{\phi^{++}}(h) \end{bmatrix} = M_\phi^2 + \mathcal{O}(h^2) = (\lambda'_+ + 4cg'^2)f^2 - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\left(\lambda + \frac{17}{12}cg'^2\right)h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \tag{B.17}$$

For h'^{0r}, h'^{0i} and h'^+ we get

$$\begin{bmatrix} m^2_{h'^{0r}}(h) \\ m^2_{h'^{0i}}(h) \\ m^2_{h^+}(h) \end{bmatrix} = M_s^2 + \mathcal{O}(h^2) = cg'^2f^2 + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\lambda h^2 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\frac{1}{6}cg'^2h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \tag{B.18}$$

[15]In writing the expansions for these masses we are assuming $cg'^2f^2 \sim \lambda h^2 \ll \lambda'_+f^2$.

From the previous expressions for the masses one can check that the cancellation of h^2 terms in $\mathrm{Str}M^2$ works except for the g'-dependent terms, as expected. The presence of the coupling g', which does not respect the $\mathrm{SU}(3)_{1,2}$ symmetries, complicates the structure of couplings in the Higgs sector. For instance, the Higgs quartic coupling after integrating out the heavy triplet is given by

$$\lambda = \frac{1}{4}\left[\lambda'_a + \lambda'_b - \frac{4}{3}cg'^2 - \frac{(\lambda'_a - \lambda'_b)^2}{(\lambda'_a + \lambda'_b + 4cg'^2)}\right], \tag{B.19}$$

to be compared with the theoretically cleaner formula (2.8) that holds in the Littlest Higgs case. All mass formulas and couplings written above reproduce those of the Littlest Higgs model in the limit $\lambda'_{a,b} \to \lambda_{a,b}$ and $g' \to 0$. After electroweak symmetry breaking some kinetic terms are non-canonical due to $\mathcal{O}(h^2/f^2)$ corrections from non-renormalizable operators. The masses above include effects from field redefinitions necessary to render canonical all fields.[16]

B.3 A Little Higgs model with T-parity

This model, proposed in [15], is also based on the SU(5)/ SO(5) structure of the Littlest Higgs model, with the same gauge and scalar field content (see appendix B.1). The gauge kinetic part of the lagrangian is as in eq. (B.8) with T-parity requiring $g_1 = g_2 = \sqrt{2}g$ and $g'_1 = g'_2 = \sqrt{2}g'$. Imposing T-invariance on the fermionic sector requires the introduction of several new degrees of freedom. Those relevant for making the fermionic lagrangian of eq. (B.10) T-symmetric are a new vector-like pair of coloured doublets $\tilde{q}_3, \tilde{q}_3^c$ (T-even) plus two new coloured singlets u'^c_T (the T-image of u'^c) and U (which is T-odd). The fermionic lagrangian reads [15]

$$\mathcal{L}_f = \frac{1}{4}\lambda_1 f \epsilon_{ijk}\epsilon_{xy}\left[(\xi Q)_i \Sigma_{jx}\Sigma_{ky}u'^c + (\tilde{\xi}Q)_i\tilde{\Sigma}_{jx}\tilde{\Sigma}_{ky}u'^c_T\right] + \lambda_2 f t' t'^c + \frac{1}{\sqrt{2}}\lambda_3 f U(u'^c - u'^c_T) + \text{h.c.}, \tag{B.20}$$

plus (heavy) mass terms for $\tilde{q}_3$. Here we have used $Q \equiv (q_3, t', \tilde{q}_3)^T$, $\xi \equiv \exp[i\Pi/f]$, $\tilde{\xi} \equiv \Omega\exp[i\Pi/f]\Omega$ [with $\Omega \equiv \mathrm{diag}(1,1,-1,1,1)$] and $\tilde{\Sigma} \equiv \tilde{\xi}^2\Sigma_0$. The index convention is as in (B.10). Finally, the scalar operators of (B.11) turn out to be given by

$$\begin{aligned}-\Delta\mathcal{L} = V \;&= 2cg^2f^4\sum_{i=1,2}\sum_a \mathrm{Tr}[(Q_i^a\Sigma)(Q_i^a\Sigma)^*] + 2cg'^2f^4\sum_{i=1,2}\mathrm{Tr}[(Y_i\Sigma)(Y_i\Sigma)^*] - \\ &-\frac{1}{16}c' f^4\lambda_1^2\epsilon^{wx}\epsilon_{yz}\left(\Sigma_{iw}\Sigma_{jx}\Sigma^{iy*}\Sigma^{jz*} + \tilde{\Sigma}_{iw}\tilde{\Sigma}_{jx}\tilde{\Sigma}^{iy*}\tilde{\Sigma}^{jz*}\right),\end{aligned} \tag{B.21}$$

which is simply a T-invariant version of (B.11).

In this model, the squared masses to $\mathcal{O}(h^2)$, needed for the calculation of the one-loop Higgs potential, are very similar to those in the Littlest Higgs model. In the gauge boson sector they are exactly the same as in (2.3), with gauge couplings related by eq. (4.1):

$$\begin{aligned}m^2_{W'}(h) &= M^2_{W'} + \mathcal{O}(h^2) = g^2f^2 - \frac{1}{4}g^2h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right), \\ m^2_{B'}(h) &= M^2_{B'} + \mathcal{O}(h^2) = \frac{1}{5}g'^2f^2 - \frac{1}{4}g'^2h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right).\end{aligned} \tag{B.22}$$

[16]An automatic way of taking care of this complication is presented in ref. [24].

In the fermion sector, the only mass relevant for our purposes is that of the heavy Top which, to order h^2, remains the same as in the Littlest Higgs model:

$$m_T^2(h) = M_T^2 + \mathcal{O}(h^2) = (\lambda_1^2 + \lambda_2^2) f^2 - \frac{1}{2}\lambda_t^2 h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \qquad \text{(B.23)}$$

The squared masses of the other heavy fermions do not have an h^2-dependence.

In the scalar sector, an important difference with respect to the Littlest Higgs model is that now there is no ϕh^2 coupling. As a result, the Higgs quartic coupling does not get modified after decoupling the triplet field and is simply given by

$$\lambda = \frac{1}{4}(\lambda_a + \lambda_b)\,, \qquad \text{(B.24)}$$

to be compared with eq. (2.8). Another direct consequence of not having a ϕh^2 coupling is the absence of the off-diagonal entries in the scalar mass matrices in the CP-even, CP-odd and charged sectors. Using the same conventions of eq. (2.10), these mass matrices are given by

$$M_\kappa^2(h) = \begin{bmatrix} a_\kappa \lambda h^2 + \mathcal{O}(h^4/f^2) & 0 \\ 0 & 4\lambda(f^2 - c_\kappa h^2) + \mathcal{O}\left(\frac{h^4}{f^2}\right) \end{bmatrix}, \qquad \text{(B.25)}$$

with the constants a_κ and c_κ exactly as in the Littlest Higgs model, eq. (2.10). The explicit masses for the different components of the heavy triplet field are still given by (B.17), and making use of (B.24) they simply read

$$\begin{bmatrix} m_{\phi^0}^2(h) \\ m_{\phi^+}^2(h) \\ m_{\phi^{++}}^2(h) \end{bmatrix} = M_\phi^2 + \mathcal{O}(h^2) = 4\lambda f^2 - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \lambda h^2 + \mathcal{O}\left(\frac{h^4}{f^2}\right). \qquad \text{(B.26)}$$

For the light eigenvalues of (B.25), which now do not mix with the triplet components, we simply get $m_{h^{0r}}^2(h) = 3\lambda h^2$, $m_{h^{0i}}^2(h) = m_{h^+}^2(h) = \lambda h^2$, as in the Standard Model.

B.4 The Simplest Little Higgs model

This is a model proposed in [16] which is based on $[\mathrm{SU}(3)\times\mathrm{U}(1)]^2/[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$, with a gauged $[\mathrm{SU}(3)\times\mathrm{U}(1)]$ subgroup broken down to the electroweak $\mathrm{SU}(2)\times\mathrm{U}(1)$. This spontaneous symmetry breaking produces 10 Goldstone bosons, 5 of which are eaten by the Higgs mechanism to make massive a complex SU(2) doublet of extra W's, $(W'^{\pm}, W'^0)$, and an extra Z'. The remaining 5 degrees of freedom are: H [an SU(2) doublet to be identified with the SM Higgs] and η (a singlet).

Explicitly, the spontaneous breaking is produced by the VEVs of two scalar triplet fields, Φ_1 and Φ_2:

$$\langle\Phi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}, \qquad \langle\Phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix}. \qquad \text{(B.27)}$$

These triplets transform under the global symmetry as

$$\Phi_1 \to e^{-i\alpha_1/3} U_1 \Phi_1\,, \qquad \Phi_2 \to e^{-i\alpha_2/3} U_2 \Phi_2\,, \qquad \text{(B.28)}$$

where U_i is an SU(3)$_i$ matrix and $e^{-i\alpha_i/3}$ are U(1)$_i$ rotations, with gauge transformations corresponding to the diagonal $U_1 = U_2$, $\alpha_1 = \alpha_2$. Using the broken generators, the Goldstone fluctuations around the vacuum (B.27) can be written as

$$\Phi_i = \exp\left[\frac{i}{f}\begin{pmatrix} 0 & 0 & h_i^+ \\ 0 & 0 & h_i^0 \\ h_i^- & h_i^{0*} & \eta_i/\sqrt{2} \end{pmatrix}\right]\begin{pmatrix} 0 \\ 0 \\ f_i \end{pmatrix}, \tag{B.29}$$

for $i = 1, 2$, with $f^2 = f_1^2 + f_2^2$. Identifying explicitly the linear combinations of h_i and η_i that correspond to the eaten Goldstones $(G^\pm, G^0, G_S)$ and the physical fields (H, η) one gets

$$\begin{aligned} \Phi_1 &= \exp\left[\frac{i}{f}\begin{pmatrix} 0 & 0 & G^+ \\ 0 & 0 & G^0 \\ G^- & G^{0*} & G_S/\sqrt{2} \end{pmatrix} + \frac{if_2}{ff_1}\begin{pmatrix} 0 & 0 & h^+ \\ 0 & 0 & h^0 \\ h^- & h^{0*} & \eta/\sqrt{2} \end{pmatrix}\right]\begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}, \\ \Phi_2 &= \exp\left[\frac{i}{f}\begin{pmatrix} 0 & 0 & G^+ \\ 0 & 0 & G^0 \\ G^- & G^{0*} & G_S/\sqrt{2} \end{pmatrix} - \frac{if_1}{ff_2}\begin{pmatrix} 0 & 0 & h^+ \\ 0 & 0 & h^0 \\ h^- & h^{0*} & \eta/\sqrt{2} \end{pmatrix}\right]\begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix}. \end{aligned}$$

The scalar kinetic part of the lagrangian is

$$\mathcal{L}_k = |D_\mu \Phi_1|^2 + |D_\mu \Phi_2|^2 \,, \tag{B.30}$$

with

$$D_\mu \Phi_i = \partial_\mu \Phi_i - igW_\mu^a T^a \Phi_i + \frac{i}{3} g_x B_\mu^x \Phi_i \,, \tag{B.31}$$

corresponding to the SU(3) $\times$ U(1)$_x$ gauged group. Obviously, g corresponds to the SU(2) gauge coupling while the relation between g, g_x and the U(1)$_Y$ gauge coupling g' is given by (5.1), which simply fixes g_x in terms of g and g'.

In order to write the one-loop Higgs potential, one can compute from (B.30) the masses of the gauge bosons in terms of $\Phi_{1,2}$. For this we find convenient to define the operator

$$\mathcal{O}_{12} \equiv \frac{1}{f^2}\left(f_1^2 f_2^2 - |\Phi_1^\dagger \Phi_2|^2\right). \tag{B.32}$$

In a background of $\langle h^0 \rangle = h/\sqrt{2}$ and η, this operator can be expanded as

$$\mathcal{O}_{12} = \frac{1}{2}h^2 - \frac{1}{48}\frac{f^2}{f_1^2 f_2^2}h^2(4h^2 + \eta^2) + \cdots \,. \tag{B.33}$$

Generically, one gets masses of the form

$$m_{H,L}^2 = \frac{M^2}{2}\left[1 \pm \sqrt{1 - 4\kappa_{12}^2 \mathcal{O}_{12}/M^2}\right], \tag{B.34}$$

where the subindices H, L stand for heavy and light masses, M is a generic mass of order f and κ_{12} is some combination of couplings. An expansion in powers of $\mathcal{O}_{12}$ gives

$$\begin{aligned} m_H^2 &= M^2 - \kappa_{12}^2 \mathcal{O}_{12} + \mathcal{O}(\mathcal{O}_{12}^2) \,, \\ m_L^2 &= \kappa_{12}^2 \mathcal{O}_{12} + \mathcal{O}(\mathcal{O}_{12}^2) \,. \end{aligned} \tag{B.35}$$

Besides the massless photon, the rest of gauge bosons have the following masses. For the charged $(W^\pm, W'^\pm)$, formula (B.34) holds with

$$M^2 = M_{W'}^2 \equiv \frac{1}{2} g^2 f^2 \,, \qquad \kappa_{12}^2 = \frac{1}{2} g^2 \,. \tag{B.36}$$

Expanding in powers of h, one reproduces (5.3). For the (Z'^0, Z^0) pair, again the masses are given by formula (B.34), now with

$$M^2 = M_{Z'}^2 \equiv \frac{2g^2}{3 - t_w^2} f^2 \,, \qquad \kappa_{12}^2 = \frac{1}{2}(g^2 + g'^2) \,, \tag{B.37}$$

where $t_w \equiv g'/g$. An expansion in powers of h reproduces (5.5). Finally, for the complex W'^0

$$m_{W'^0}^2 = M_{W'}^2 \,, \qquad \kappa_{12}^2 = 0 \,. \tag{B.38}$$

In the fermion sector, the Yukawa part of the lagrangian, reads

$$\mathcal{L}_Y = \lambda_1 u_1^c \Phi_1^\dagger \Psi_Q + \lambda_2 u_2^c \Phi_2^\dagger \Psi_Q + \text{h.c.} \,, \tag{B.39}$$

with generation indices suppressed (we only care about the third family). Here Ψ_Q is an SU(3) triplet (with x-charge 1/3) that contains the usual quark doublet while $u_{1,2}^c$ are SU(3) singlets (with x-charge $-2/3$). A combination of u_1^c and u_2^c corresponds to the SM top quark field while the orthogonal combination gets a heavy mass with the third component of Ψ_Q. The explicit masses of these fields follow the pattern of (B.34) with

$$M^2 = M_T^2 \equiv \lambda_1^2 f_1^2 + \lambda_2^2 f_2^2 \,, \qquad \kappa_{12}^2 = \lambda_t^2 \,, \tag{B.40}$$

where λ_t is the SM top Yukawa coupling, given by

$$\frac{f^2}{\lambda_t^2} = \frac{f_1^2}{\lambda_2^2} + \frac{f_2^2}{\lambda_1^2} \,. \tag{B.41}$$

An expansion in powers of h gives (5.8).

From the generic formula for the masses in eq. (B.34) one sees that $\mathrm{Str} M^2$ is field independent. Therefore, and in contrast with previous models, one-loop quadratically divergent corrections from gauge or fermion loops do not induce scalar operators to be added to the lagrangian. (This is not the case for scalars, see section 5).

Less divergent one-loop corrections do induce both a mass term and a quartic coupling for the Higgs, as explicitly shown in the main text. Here we present the one-loop potential in terms of the fields Φ_1 and Φ_2. In the $\overline{\text{MS}}$ scheme with the renormalization scale set to $Q = \Lambda$, it is straightforward to compute the one-loop potential including fermion and gauge boson loops, once the masses are known as functions of $\mathcal{O}_{12}$. Performing an expansion in powers of $\mathcal{O}_{12}$, this potential reads

$$V = \delta m^2 \mathcal{O}_{12} + \delta_1 \lambda(\mathcal{O}_{12}) \mathcal{O}_{12}^2 + \cdots \tag{B.42}$$

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with δm^2 as given in (5.12) and $\delta_1\lambda(\mathcal{O}_{12})$ as given by (5.13) with the h-dependence coming through the dependence of the masses on $\mathcal{O}_{12}$, see eq. (B.34). Expanding further in powers of h and η, we get

$$V(h) = \frac{1}{2}\delta m^2 h^2 + \frac{1}{4}\left[\delta_1\lambda(h) - \frac{\delta m^2}{3}\frac{f^2}{f_1^2 f_2^2}\right] h^4 - \frac{\delta m^2}{48}\frac{f^2}{f_1^2 f_2^2} h^2\eta^2 + \cdots \tag{B.43}$$

which reproduces (5.11) and gives also the η terms.

Finally, a mass operator is introduced in the tree level potential to get a correct electroweak symmetry breaking [16]

$$\delta_0 V = \mu^2 \mathcal{O}_X \equiv \mu^2(2f_1 f_2 - \Phi_1^\dagger\Phi_2 - \Phi_2^\dagger\Phi_1)\,, \tag{B.44}$$

which, in terms of h and η, gives

$$\delta_0 V = \frac{1}{2}\mu_0^2(h^2+\eta^2) - \frac{1}{48}\frac{\mu_0^2 f^2}{f_1^2 f_2^2}(h^4 + 3h^2\eta^2 + \eta^4) + \cdots \tag{B.45}$$

with $\mu_0^2 \equiv \mu^2 f^2/(f_1 f_2)$.

As explained in the main text, by choosing $\mu_0^2 > 0$ we get a positive contribution to the Higgs mass parameter that can compensate the heavy Top contribution in δm^2. The tree-level value of the Higgs quartic coupling λ from (B.45) is then negative but the large (and positive) radiative corrections to λ can easily overcome this effect. We also see that (B.45) gives a mass of order μ_0 to the η field [this field had no mass term in (B.43)].

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